دانشکده مهندسی مکانیک

درس دینامیک

Engineering Mechanics Dynamics

Meriam, Kraige & Bolton

9th Edition

Chapter 5: Plane Kinetics of Rigid Bodies

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- Chapter 2: Kinematics of Particles
- Chapter 3: Kinetics of Particles
- Chapter 4: Kinetics of Systems of Particles
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- → Chapter 6: **Plane Kinetics of Rigid Bodies**

CHAPTER 6

Plane Kinetics of Rigid Bodies

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The principles of this chapter must be applied during the design of the massive blades of large wind turbines.

CHAPTER OUTLINE

6/1 Introduction

SECTION A Force, Mass, and Acceleration

6/2 General Equations of Motion
6/3 Translation
6/4 Fixed-Axis Rotation
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SECTION B Work and Energy

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SECTION C Impulse and Momentum

6/8 Impulse-Momentum Equations
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6/1 Introduction

- The kinetics of rigid bodies treats the relationships between the external forces acting on a body and the corresponding translational and rotational motions of the body

- For our purpose, a body which can be approximated as a thin slab with its motion confined to the plane of the slab will be considered to be in plane motion.

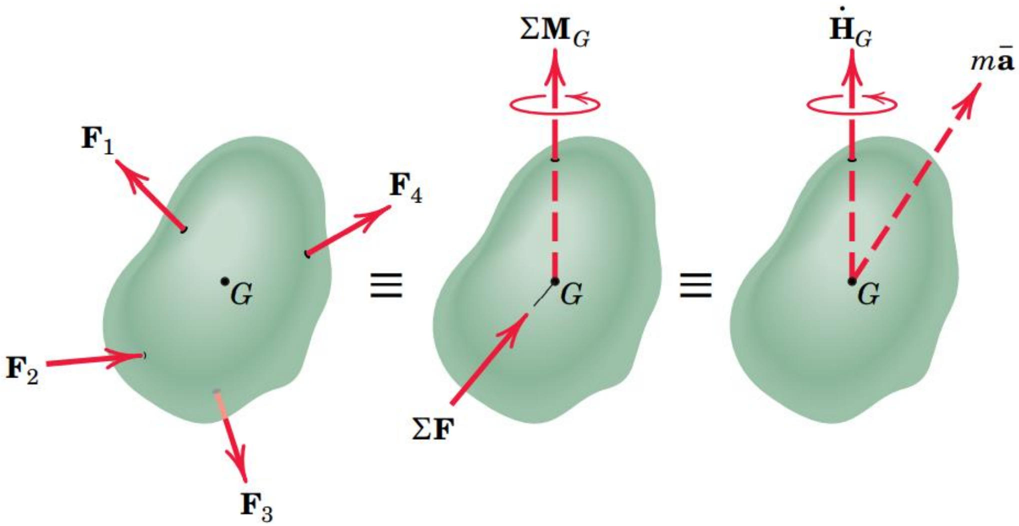
  - ✓Section A: forces and moments to instantaneous linear and angular accelerations relations
  - ✓Section B: method of work and energy
  - ✓Section C: methods of impulse and momentum

SECTION A Force, Mass, and Acceleration

6/2 General Equations of Motion

- The force equation: $\Sigma \mathbf{F} = m\bar{\mathbf{a}}$

- The moment equation taken about the mass center: $\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G$



Free-Body Diagram | Equivalent Force-Couple System | Kinetic Diagram

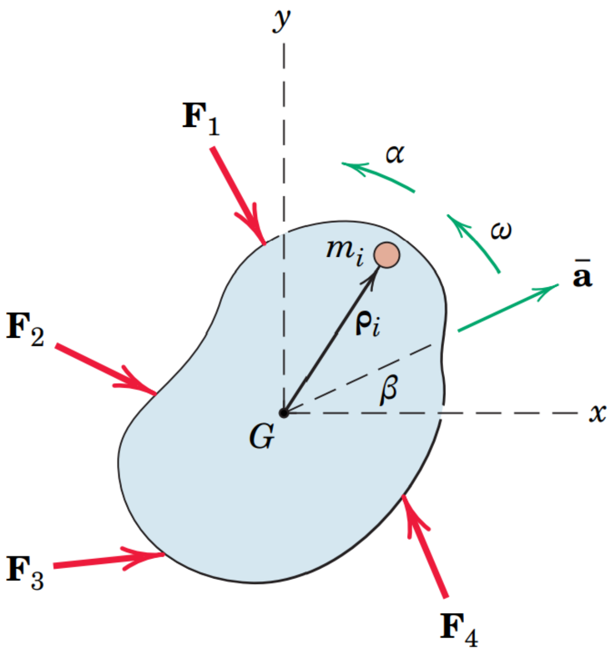
6/2 General Equations of Motion

Plane-Motion Equations

- A rigid body moving with plane motion in the x-y plane
  - The mass moment of inertia:

$$H_G = \Sigma \rho_i^2 m_i \omega = \omega \Sigma \rho_i^2 m_i \rightarrow \int \rho^2 \, dm \rightarrow \bar{I}$$

$$H_G = \bar{I}\omega \rightarrow \Sigma M_G = \dot{H}_G = \bar{I}\dot{\omega} = \bar{I}\alpha$$



$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

$$\Sigma M_G = \bar{I}\alpha$$

using x-y, n-t, or r-θ coordinates

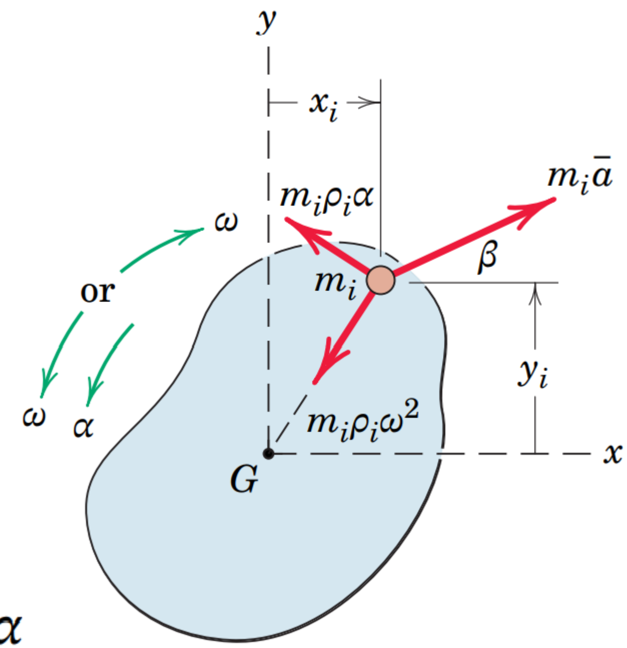
6/2 General Equations of Motion

Alternative Derivation

- By referring directly to the forces which act on the representative particle of mass m_i

$$M_{G_i} = m_i\rho_i^2\alpha + (m_i\bar{a}\sin\beta)x_i - (m_i\bar{a}\cos\beta)y_i$$

$$\Sigma M_G = \Sigma m_i\rho_i^2\alpha + \bar{a}\sin\beta\,\Sigma m_i x_i - \bar{a}\cos\beta\,\Sigma m_i y_i$$



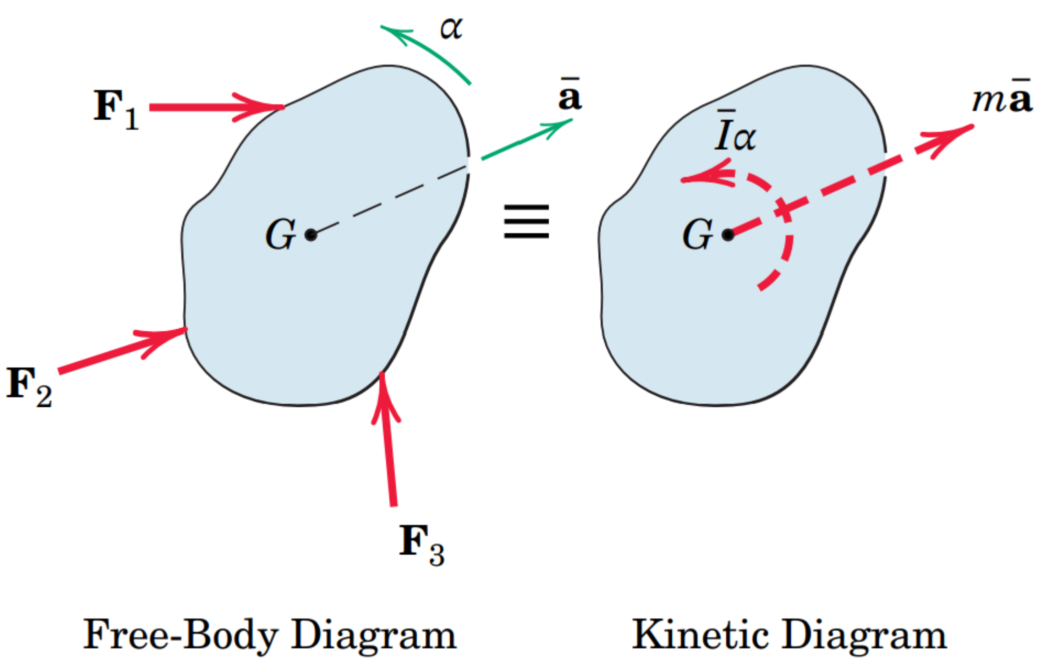
$$\Sigma m_i x_i = m\bar{x} = 0$$

$$\Sigma m_i y_i = m\bar{y} = 0$$

$$\Sigma M_G = \Sigma m_i\rho_i^2\alpha = \bar{I}\alpha$$

6/2 General Equations of Motion

- ✓ The free-body diagram discloses the forces and moments appearing on the lefthand side of equations of motion.
- ✓ The kinetic diagram discloses the resulting dynamic response in terms of the translational term and the rotational term which appear on the right-hand side of equations of motion.



6/2 General Equations of Motion

Alternative Moment Equations

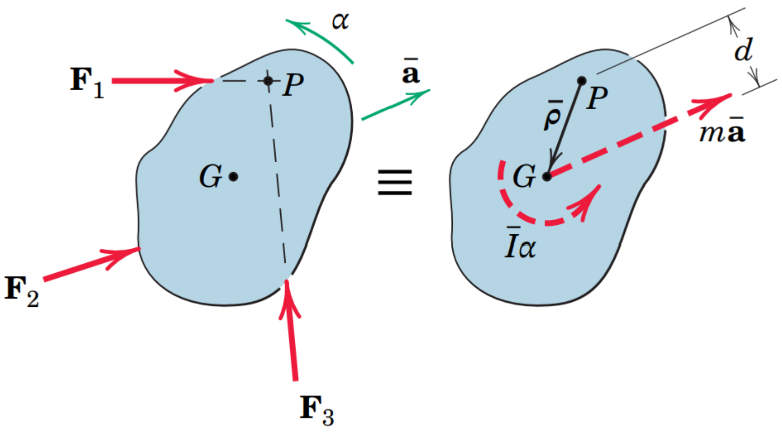
- General equation for moments about an arbitrary point P

$$\Sigma \mathbf{M}_P = \dot{\mathbf{H}}_G + \bar{\boldsymbol{\rho}} \times m\bar{\mathbf{a}}$$

$$\Sigma M_P = \bar{I}\alpha + m\bar{a}d$$

$$\Sigma \mathbf{M}_P = (\dot{\mathbf{H}}_P)_{\text{rel}} + \bar{\boldsymbol{\rho}} \times m\mathbf{a}_P$$

$$\Sigma \mathbf{M}_P = I_P\boldsymbol{\alpha} + \bar{\boldsymbol{\rho}} \times m\mathbf{a}_P$$



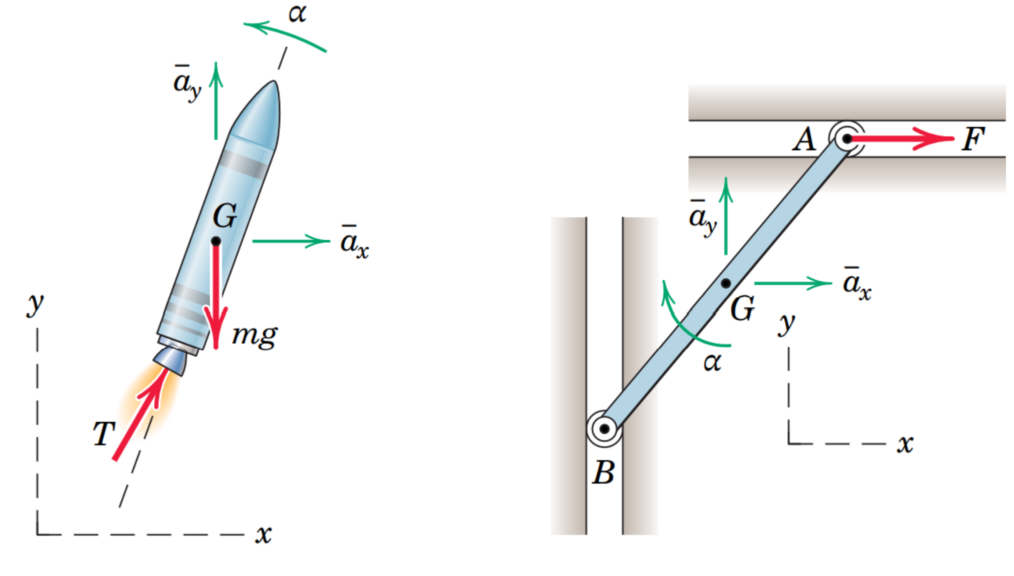
- When point P becomes a point O fixed in an inertial reference system

$$\Sigma M_O = I_O\alpha$$

6/2 General Equations of Motion

Unconstrained and Constrained Motion

- The motion of a rigid body may be unconstrained or constrained.
- The two components a_x and a_y of the mass center acceleration and the angular acceleration α may be determined independently or not.
- In general, dynamics problems which involve physical constraints to motion require a kinematic analysis relating linear to angular acceleration before the force and moment equations of motion can be solved.



(*a*) Unconstrained Motion (*b*) Constrained Motion

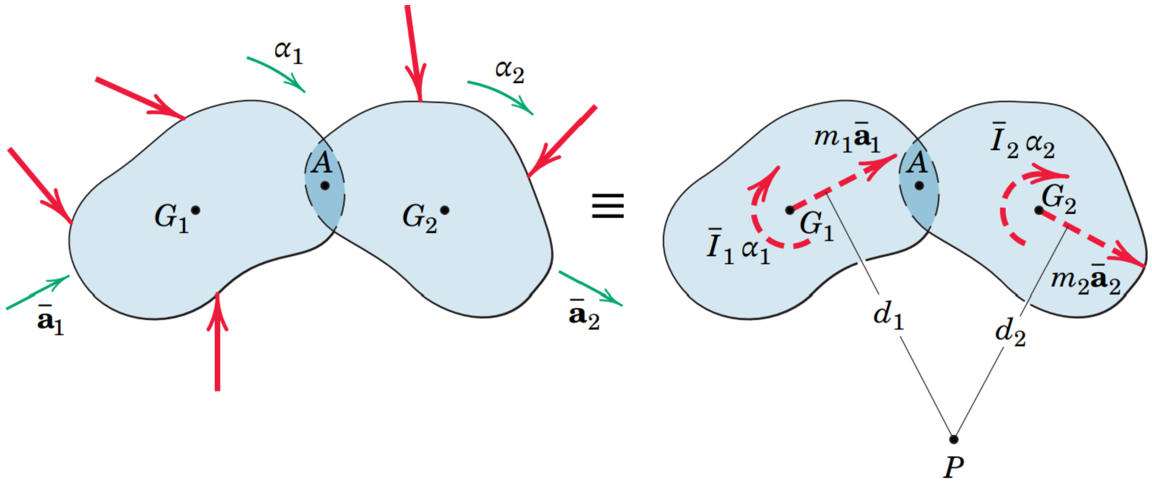
6/2 General Equations of Motion

Systems of Interconnected Bodies

- ✓ In problems dealing with two or more connected rigid bodies whose motions are related kinematically, it is convenient to analyze the bodies as an entire system.

$$\Sigma \mathbf{F} = \Sigma m\bar{\mathbf{a}}$$

$$\Sigma M_P = \Sigma \bar{I}\alpha + \Sigma m\bar{a}d$$



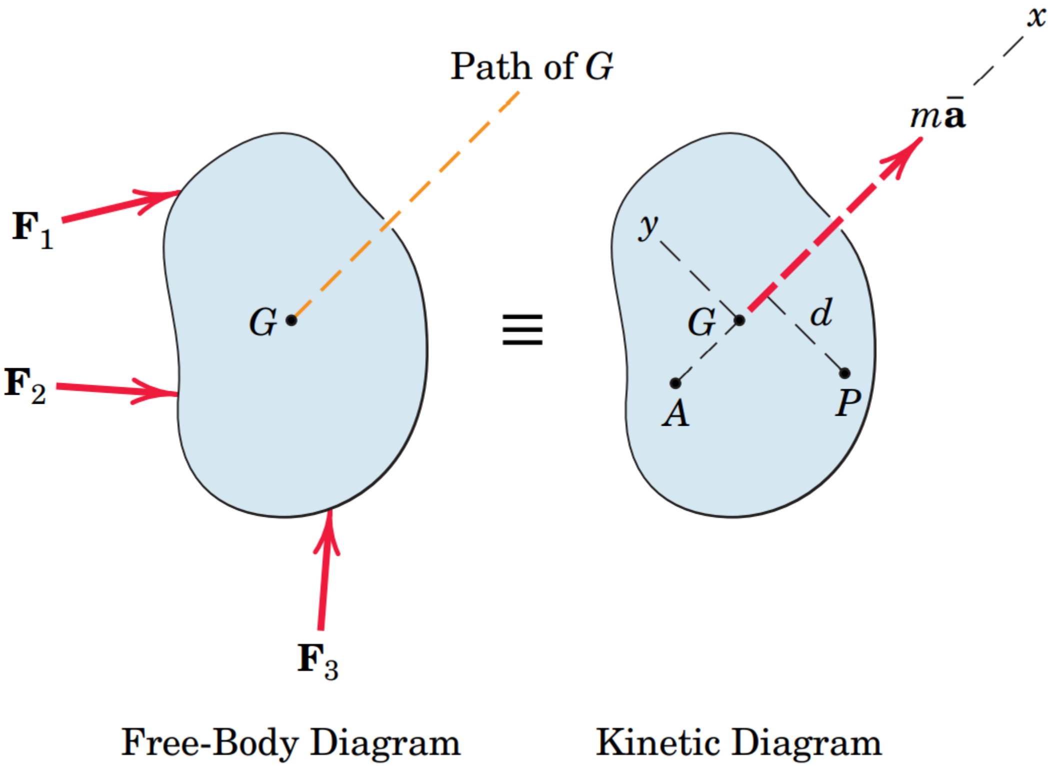
Free-Body Diagram of System ≡ Kinetic Diagram of System

- ✓ If there are more than three remaining unknowns, the E.O.M. is not sufficient to solve the problem.

6/3 Translation

- There is no angular motion of the translating body, so that both ω and α are zero.
  - Rectilinear Translation:

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$
$$\Sigma M_G = \bar{I}\alpha = 0$$



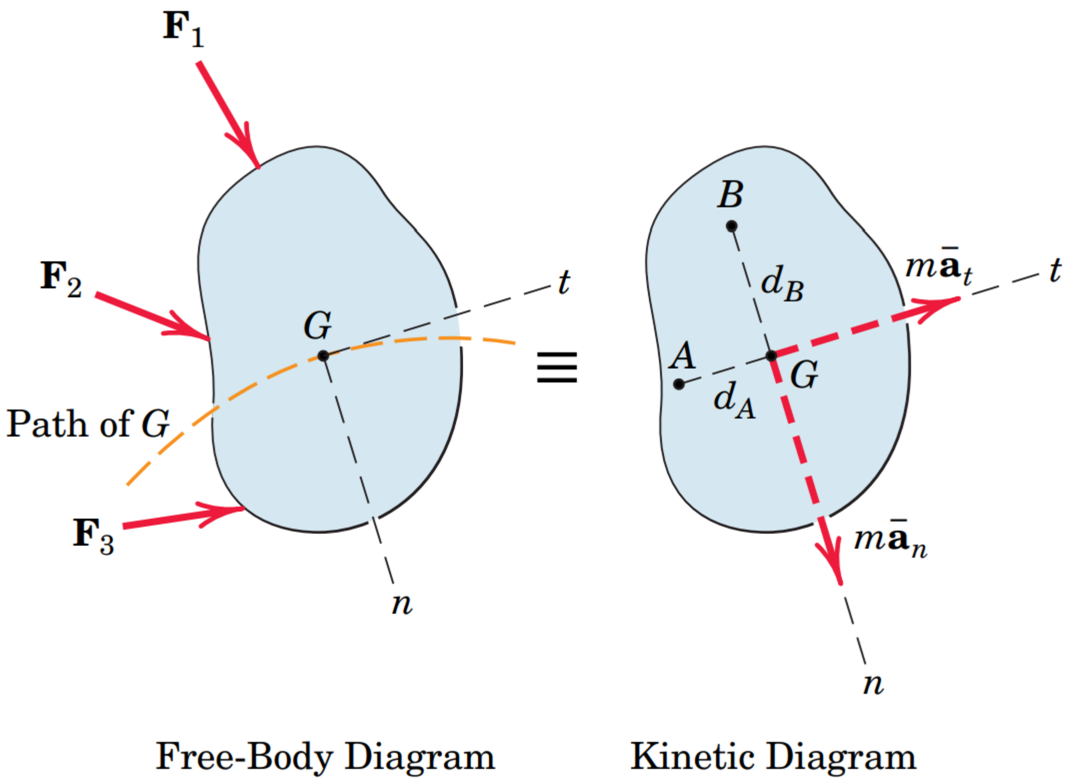
(a) Rectilinear Translation
($\alpha = 0$, $\omega = 0$)

6/3 Translation

- There is no angular motion of the translating body, so that both ω and α are zero.
  - Curvilinear Translation:

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

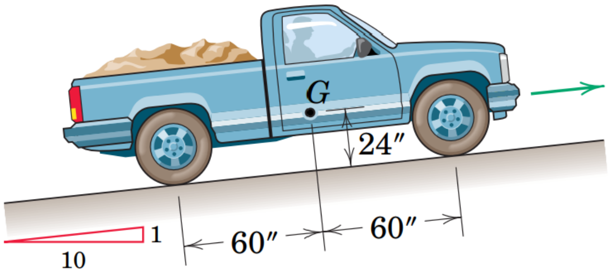
$$\Sigma M_G = \bar{I}\alpha = 0$$



(b) Curvilinear Translation
$(\alpha = 0, \omega = 0)$

SAMPLE PROBLEM 6/1

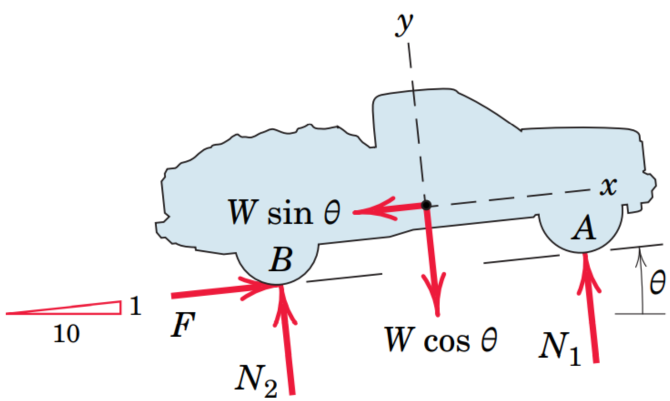
The pickup truck weighs 3220 lb and reaches a speed of 30 mi/hr from rest in a distance of 200 ft up the 10-percent incline with constant acceleration. Calculate the normal force under each pair of wheels and the friction force under the rear driving wheels. The effective coefficient of friction between the tires and the road is known to be at least 0.80.

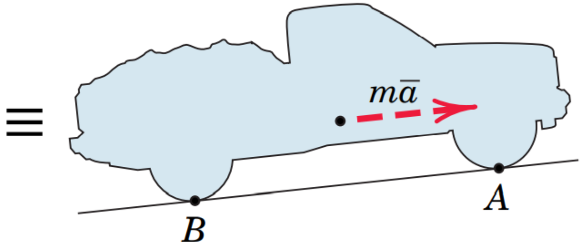


$$m\bar{a} = \frac{3220}{32.2}(4.84) = 484 \text{ lb}$$

$[\Sigma F_x = m\bar{a}_x]$	$F - 320 = 484$	$F = 804$ lb
$[\Sigma F_y = m\bar{a}_y = 0]$	$N_1 + N_2 - 3200 = 0$	
$[\Sigma M_G = \bar{I}\alpha = 0]$	$60N_1 + 804(24) - N_2(60) = 0$	

$$N_1 = 1441 \text{ lb} \qquad N_2 = 1763 \text{ lb}$$





SAMPLE PROBLEM 6/1

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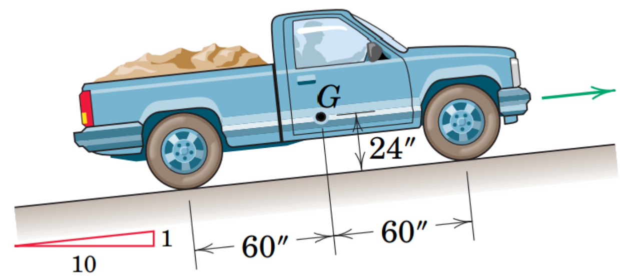


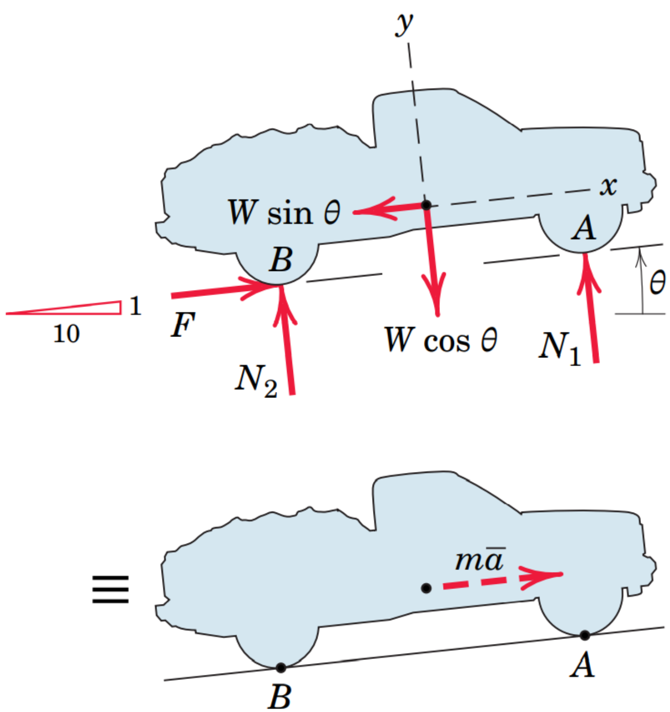
Alternative Solution

$$[\Sigma M_A = m\bar{a}d] \qquad 120N_2 - 60(3200) - 24(320) = 484(24)$$

$$N_2 = 1763 \text{ lb}$$

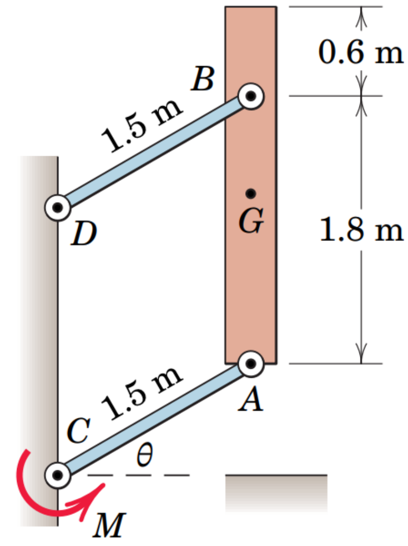
$$[\Sigma M_B = m\bar{a}d] \qquad 3200(60) - 320(24) - 120N_1 = 484(24)$$

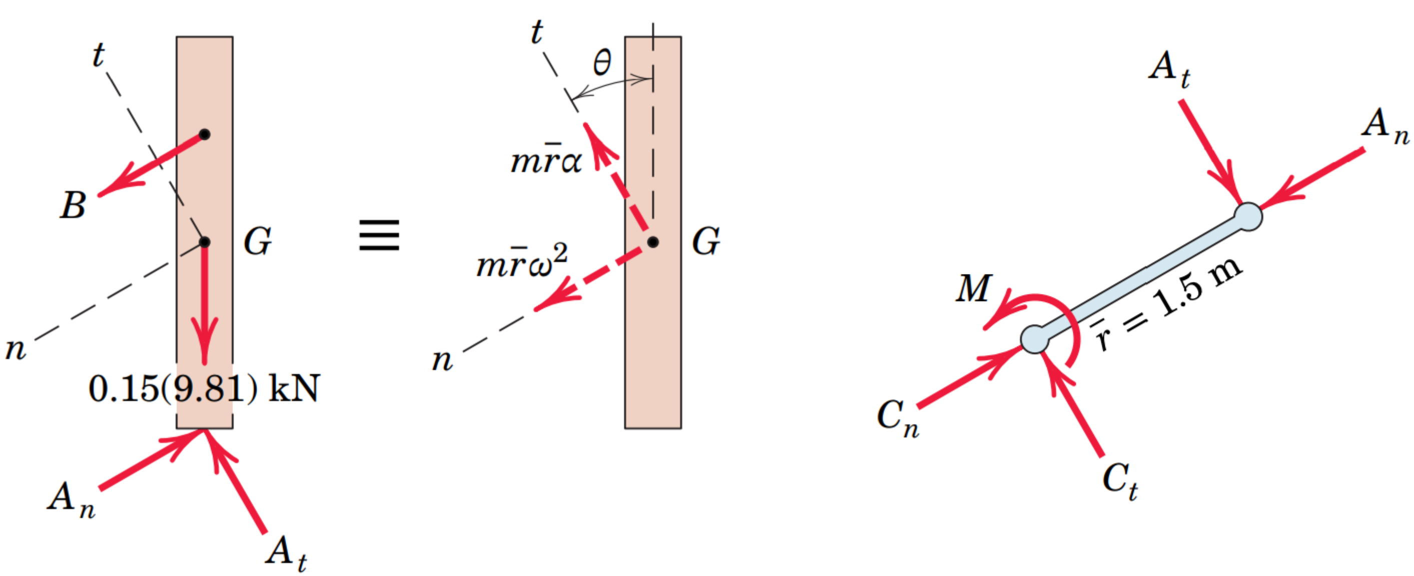
$$N_1 = 1441 \text{ lb}$$



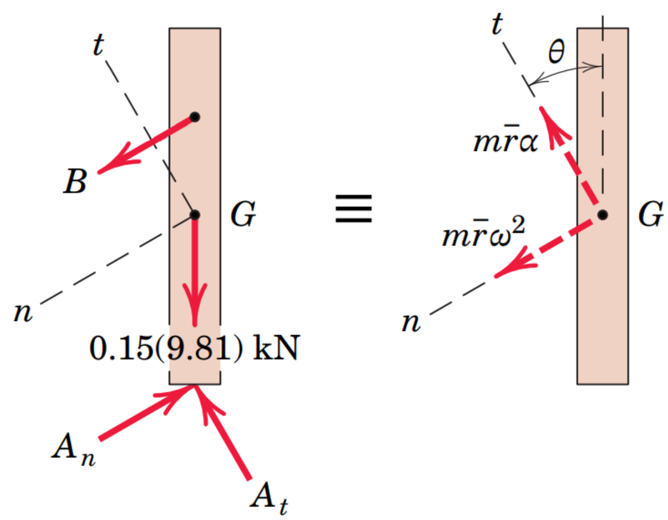
SAMPLE PROBLEM 6/2

The vertical bar AB has a mass of 150 kg with center of mass G midway between the ends. The bar is elevated from rest at $\theta = 0$ by means of the parallel links of negligible mass, with a constant couple $M = 5$ kN·m applied to the lower link at C. Determine the angular acceleration α of the links as a function of θ and find the force B in the link DB at the instant when $\theta = 30°$.





$$A_t = M/\overline{AC} = 5/1.5 = 3.33 \text{ kN}$$



$[\Sigma F_t = m\bar{a}_t]$ $$3.33 - 0.15(9.81)\cos\theta = 0.15(1.5\alpha)$$

$$\alpha = 14.81 - 6.54\cos\theta \text{ rad/s}^2$$

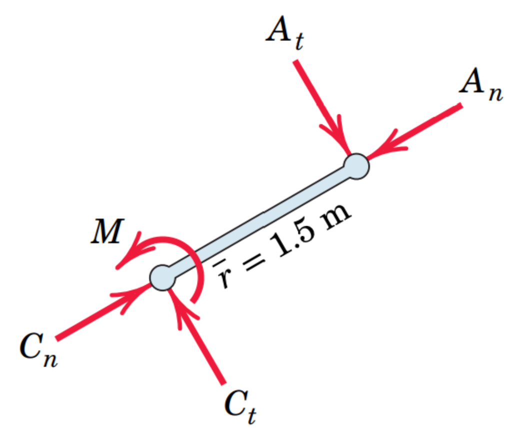
$[\omega\, d\omega = \alpha\, d\theta]$ $$\int_0^\omega \omega\, d\omega = \int_0^\theta (14.81 - 6.54\cos\theta)\, d\theta$$

$$\omega^2 = 29.6\theta - 13.08\sin\theta$$

$$(\omega^2)_{30^\circ} = 8.97 \text{ (rad/s)}^2 \qquad \alpha_{30^\circ} = 9.15 \text{ rad/s}^2$$

$$m\bar{r}\omega^2 = 0.15(1.5)(8.97) = 2.02 \text{ kN}$$

$$m\bar{r}\alpha = 0.15(1.5)(9.15) = 2.06 \text{ kN}$$



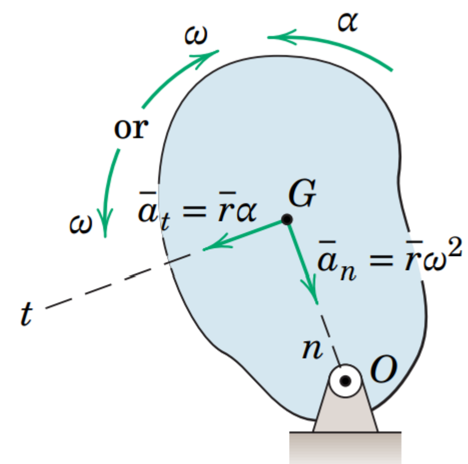
$[\Sigma M_A = m\bar{a}d]$ $$1.8\cos 30^\circ\, B = 2.02(1.2)\cos 30^\circ + 2.06(0.6)$$

$$B = 2.14 \text{ kN}$$

6/4 Fixed-Axis Rotation

- All points in the body describe circles about the rotation axis, and all lines of the body in the plane of motion have the same angular velocity ω and angular acceleration α.

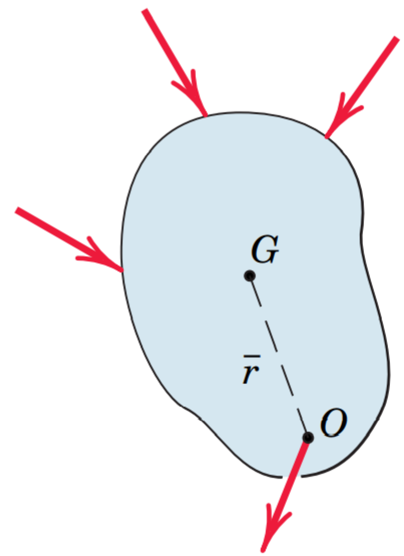
$$\Sigma M_O = \bar{I}\alpha + m\bar{a}_t\bar{r} \rightarrow I_O = \bar{I} + m\bar{r}^2 \rightarrow \Sigma M_O = (I_O - m\bar{r}^2)\alpha + m\bar{r}^2\alpha = I_O\alpha$$



Fixed-Axis Rotation

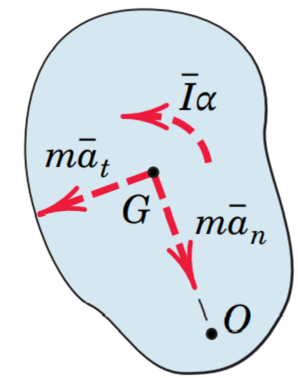
$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$
$$\Sigma M_G = \bar{I}\alpha$$

$$\Sigma M_O = I_O\alpha$$



Free-Body Diagram

≡



Kinetic Diagram

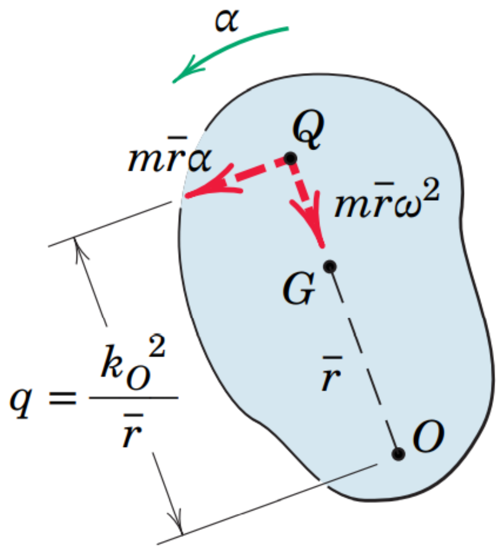
6/4 Fixed-Axis Rotation

Center of Percussion

- We may combine the resultant-force component mat and resultant couple Iα by moving mat to a parallel position through point Q on line OG.

$$m\bar{r}\alpha q = \bar{I}\alpha + m\bar{r}\alpha(\bar{r})$$

$$I_O = k_O{}^2 m \quad \rightarrow \quad q = k_O{}^2/\bar{r}$$



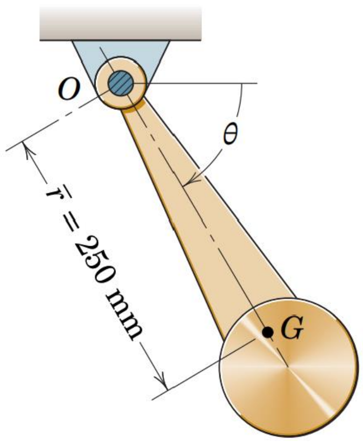
- Point Q is called the center of percussion and has the unique property that the resultant of all forces applied to the body must pass through it.

- It follows that the sum of the moments of all forces about the center of percussion is always zero

$$\Sigma M_Q = 0$$

SAMPLE PROBLEM 6/4

The pendulum has a mass of 7.5 kg with center of mass at G and has a radius of gyration about the pivot O of 295 mm. If the pendulum is released from rest at $\theta = 0$, determine the total force supported by the bearing at the instant when $\theta = 60°$. Friction in the bearing is negligible.

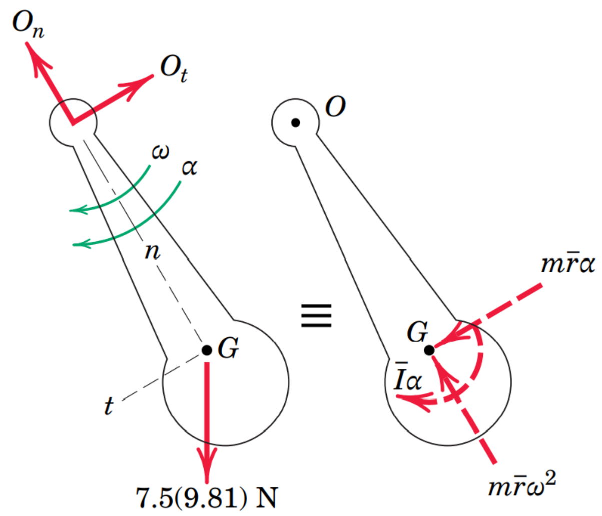


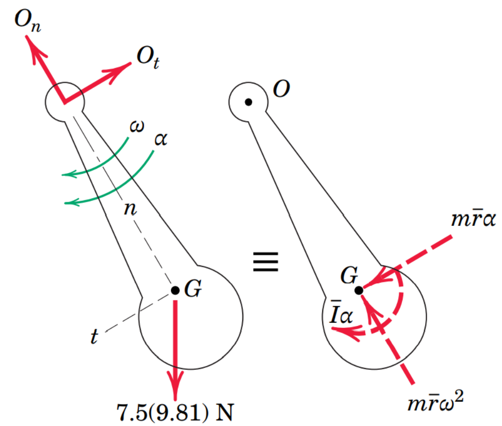
$[\Sigma M_O = I_O\alpha]$ $$7.5(9.81)(0.25)\cos\theta = (0.295)^2(7.5)\alpha$$

$$\alpha = 28.2 \cos\theta \text{ rad/s}^2$$

$[\omega\, d\omega = \alpha\, d\theta]$ $$\int_0^{\omega} \omega\, d\omega = \int_0^{\pi/3} 28.2 \cos\theta\, d\theta$$

$$\omega^2 = 48.8 \text{ (rad/s)}^2$$





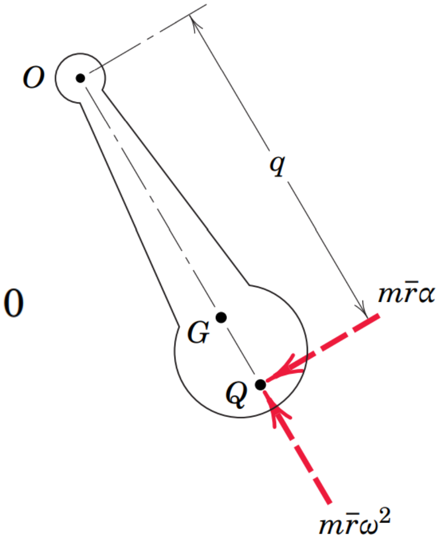
$[\Sigma F_n = m\bar{r}\omega^2]$ $\qquad O_n - 7.5(9.81)\sin 60° = 7.5(0.25)(48.8)$

$$O_n = 155.2 \text{ N}$$

$[\Sigma F_t = m\bar{r}\alpha]$ $\qquad -O_t + 7.5(9.81)\cos 60° = 7.5(0.25)(28.2)\cos 60°$

$$O_t = 10.37 \text{ N}$$

$$O = \sqrt{(155.2)^2 + (10.37)^2} = 155.6 \text{ N}$$



$[q = k_O^2/\bar{r}]$ $\qquad q = \dfrac{(0.295)^2}{0.250} = 0.348 \text{ m}$

$[\Sigma M_Q = 0]$ $\qquad O_t(0.348) - 7.5(9.81)(\cos 60°)(0.348 - 0.250) = 0$

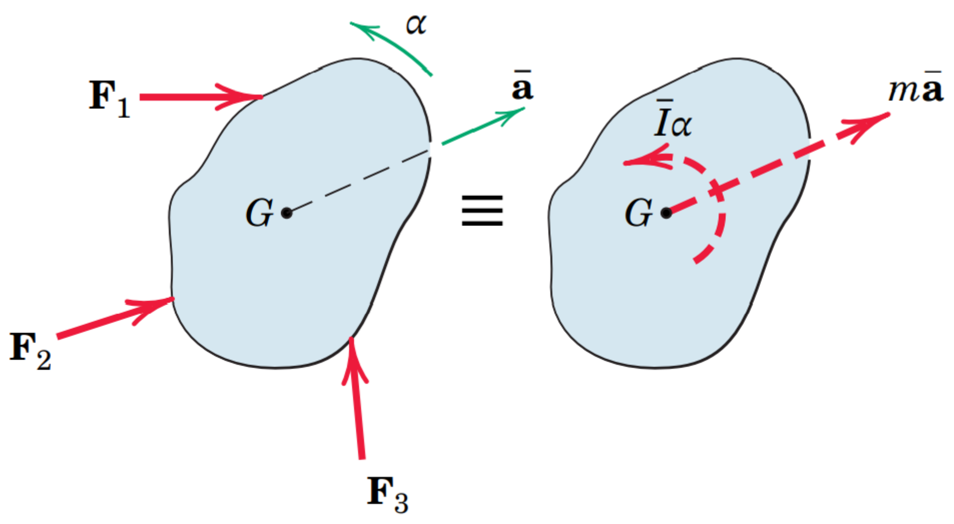
$$O_t = 10.37 \text{ N}$$

6/5 General Plane Motion

- The dynamics of a rigid body in general plane motion combines translation and rotation.

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}}$$

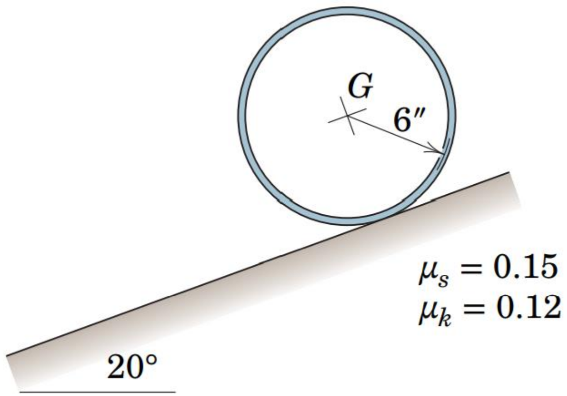
$$\Sigma M_G = \bar{I}\alpha$$



Free-Body Diagram Kinetic Diagram

SAMPLE PROBLEM 6/5

A metal hoop with a radius $r = 6$ in. is released from rest on the 20° incline. If the coefficients of static and kinetic friction are $\mu_s = 0.15$ and $\mu_k = 0.12$, determine the angular acceleration α of the hoop and the time t for the hoop to move a distance of 10 ft down the incline.



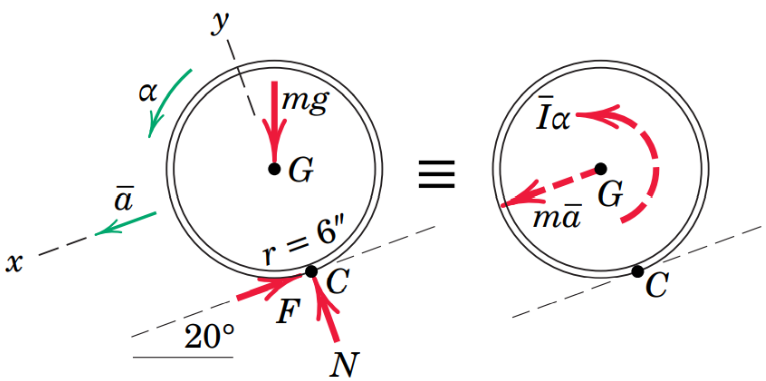
Assume that the hoop rolls without slipping

$\bar{a} = r\alpha$

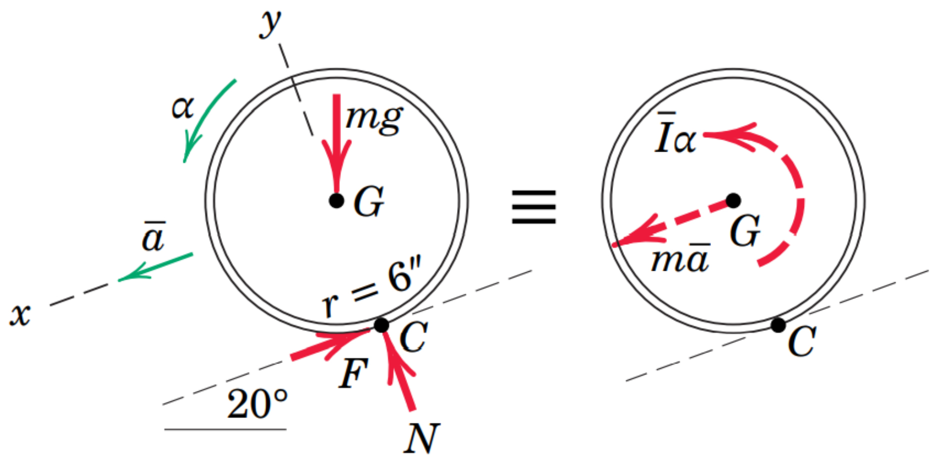
$[\Sigma F_x = m\bar{a}_x]$ $\quad mg \sin 20° - F = m\bar{a}$

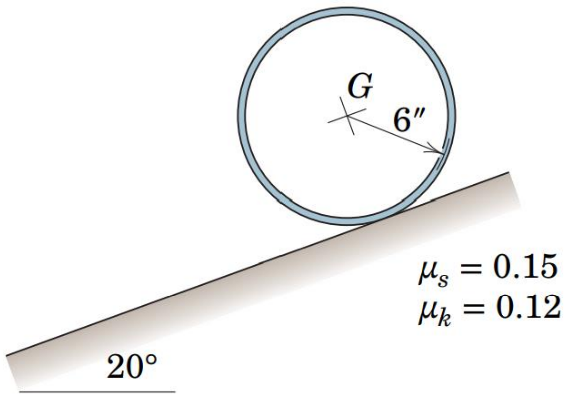
$[\Sigma F_y = m\bar{a}_y = 0]$ $\quad N - mg \cos 20° = 0$

$[\Sigma M_G = \bar{I}\alpha]$ $\quad Fr = mr^2\alpha$



$$\bar{a} = \frac{g}{2}\sin 20° = \frac{32.2}{2}(0.342) = 5.51 \text{ ft/sec}^2$$





Alternatively: $[\Sigma M_C = \bar{I}\alpha + m\bar{a}d]$ $\quad mgr \sin 20^\circ = mr^2 \dfrac{\bar{a}}{r} + m\bar{a}r$ $\quad \bar{a} = \dfrac{g}{2} \sin 20^\circ$

$$F = mg \sin 20^\circ - m\frac{g}{2} \sin 20^\circ = 0.1710mg$$

$$N = mg \cos 20^\circ = 0.940mg$$

→ hoop slips as it rolls

$[F_{max} = \mu_s N]$ $\quad F_{max} = 0.15(0.940mg) = 0.1410mg$

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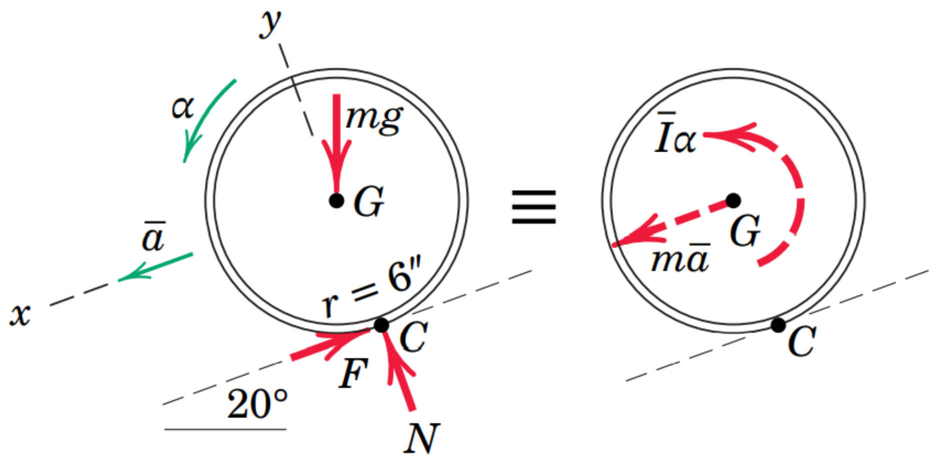
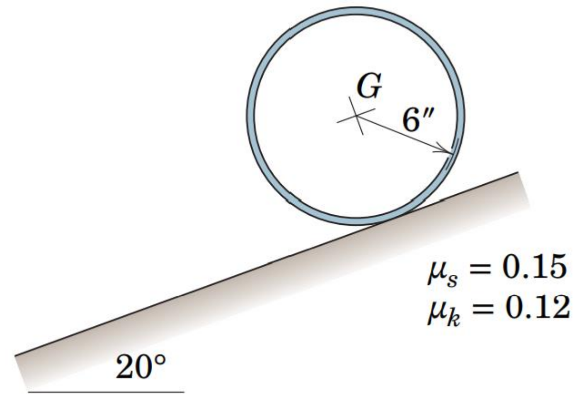




$[F = \mu_k N]$ $\qquad F = 0.12(0.940mg) = 0.1128mg$

$[\Sigma F_x = m\bar{a}_x]$ $\qquad mg \sin 20° - 0.1128mg = m\bar{a}$

$$\bar{a} = 0.229(32.2) = 7.38 \text{ ft/sec}^2$$

$[\Sigma M_G = \bar{I}\alpha]$ $\qquad 0.1128mg(r) = mr^2\alpha$

$$\alpha = \frac{0.1128(32.2)}{6/12} = 7.26 \text{ rad/sec}^2$$

$[x = \frac{1}{2}at^2]$ $\rightarrow$ $t = \sqrt{\frac{2x}{a}} = \sqrt{\frac{2(10)}{7.38}} = 1.646 \text{ sec}$

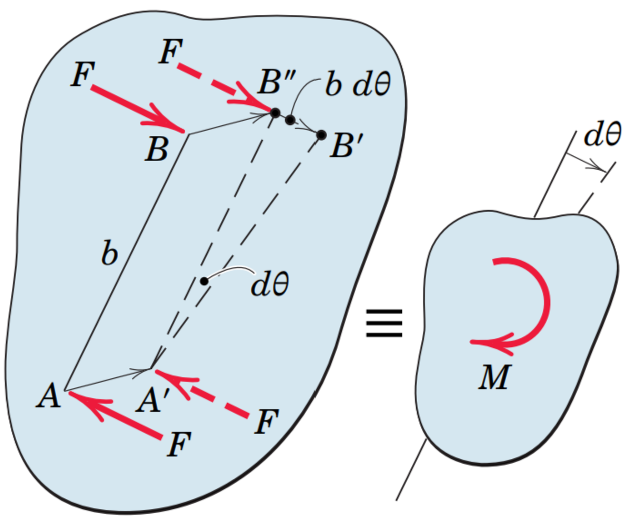
SECTION B Work and Energy

6/6 Work-Energy Relations

Work of Forces and Couples

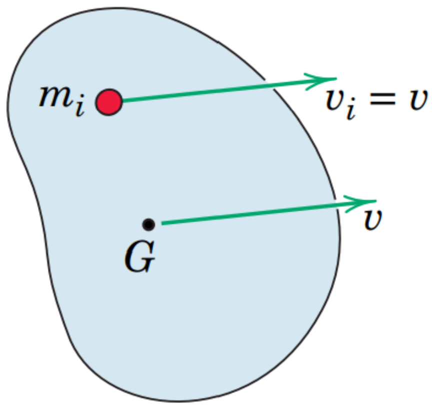
$$U = \int \mathbf{F} \cdot d\mathbf{r} \qquad \text{or} \qquad U = \int (F \cos \alpha)\, ds$$

$$U = \int M\, d\theta$$

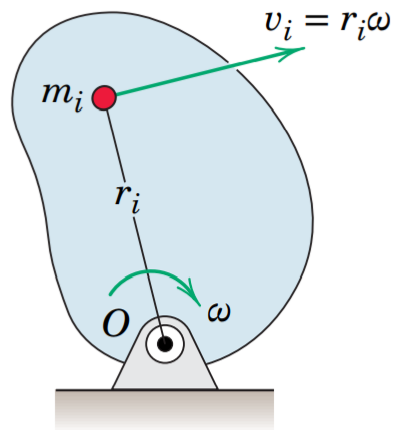


6/6 Work-Energy Relations

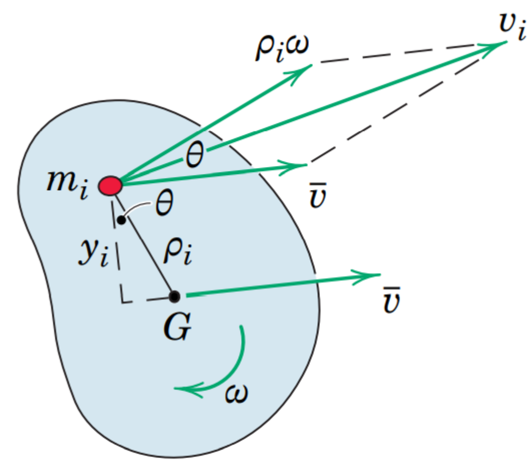
Kinetic Energy



(a) Translation



(b) Fixed-Axis Rotation



(c) General Plane Motion

$$T = \frac{1}{2}mv^2$$

$$T = \frac{1}{2}I_O\omega^2$$

$$T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2$$

$$T = \frac{1}{2}I_C\omega^2$$

C: instantaneous center of zero velocity

6/6 Work-Energy Relations

Potential Energy and the Work-Energy Equation

- Gravitational potential energy V_g and elastic potential energy V_e

$$T_1 + U_{1\text{-}2} = T_2$$

$$T_1 + V_1 + U'_{1\text{-}2} = T_2 + V_2$$

Power

$$P = \frac{dU}{dt} = \frac{\mathbf{F}\cdot d\mathbf{r}}{dt} = \mathbf{F}\cdot\mathbf{v}$$

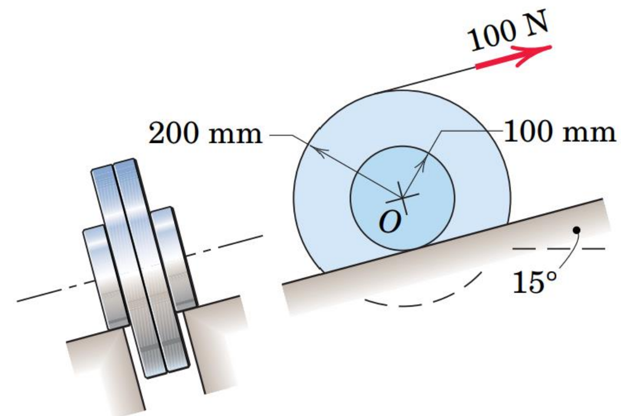
$$P = \frac{dU}{dt} = \frac{M\,d\theta}{dt} = M\omega$$

→

$$P = \mathbf{F}\cdot\mathbf{v} + M\omega$$

SAMPLE PROBLEM 6/9

The wheel rolls up the incline on its hubs without slipping and is pulled by the 100-N force applied to the cord wrapped around its outer rim. If the wheel starts from rest, compute its angular velocity ω after its center has moved a distance of 3 m up the incline. The wheel has a mass of 40 kg with center of mass at O and has a centroidal radius of gyration of 150 mm. Determine the power input from the 100-N force at the end of the 3-m motion interval.



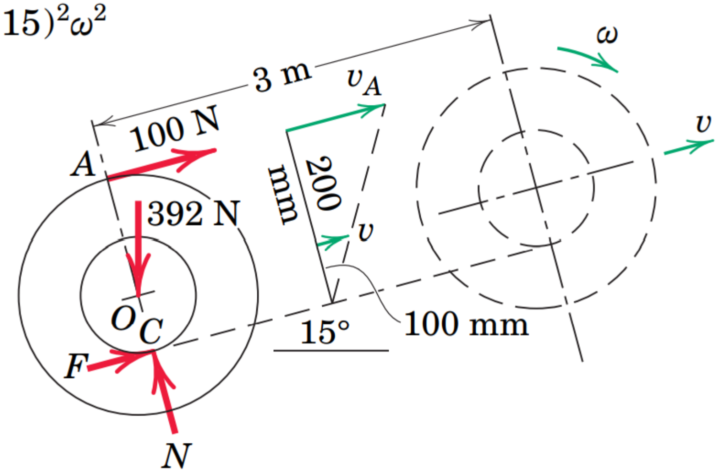
$$U_{1\text{-}2} = 100\,\frac{200 + 100}{100}(3) - (392 \sin 15°)(3) = 595 \text{ J}$$

$$\left[T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2\right] \qquad T_1 = 0 \qquad T_2 = \tfrac{1}{2}40(0.10\omega)^2 + \tfrac{1}{2}40(0.15)^2\omega^2 = 0.650\omega^2$$

$$\left[T = \tfrac{1}{2}I_C\omega^2\right] \qquad T = \tfrac{1}{2}40[(0.15)^2 + (0.10)^2]\omega^2 = 0.650\omega^2$$

$$[T_1 + U_{1\text{-}2} = T_2] \qquad 0 + 595 = 0.650\omega^2 \qquad \omega = 30.3 \text{ rad/s}$$

$$[P = \mathbf{F}\cdot\mathbf{v}] \qquad P_{100} = 100(0.3)(30.3) = 908 \text{ W}$$



SECTION C Impulse and Momentum

6/8 Impulse-Momentum Equations

Linear Momentum

$$\mathbf{G} = m\bar{\mathbf{v}}$$

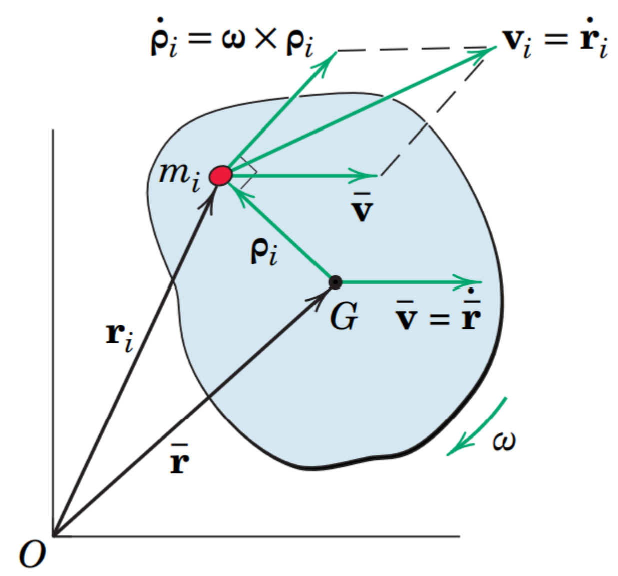
$$\Sigma \mathbf{F} = \dot{\mathbf{G}}$$

$$\mathbf{G}_1 + \int_{t_1}^{t_2} \Sigma \mathbf{F}\, dt = \mathbf{G}_2$$

$$\Sigma F_x = \dot{G}_x \qquad \Sigma F_y = \dot{G}_y$$

$$(G_x)_1 + \int_{t_1}^{t_2} \Sigma F_x\, dt = (G_x)_2$$

$$(G_y)_1 + \int_{t_1}^{t_2} \Sigma F_y\, dt = (G_y)_2$$



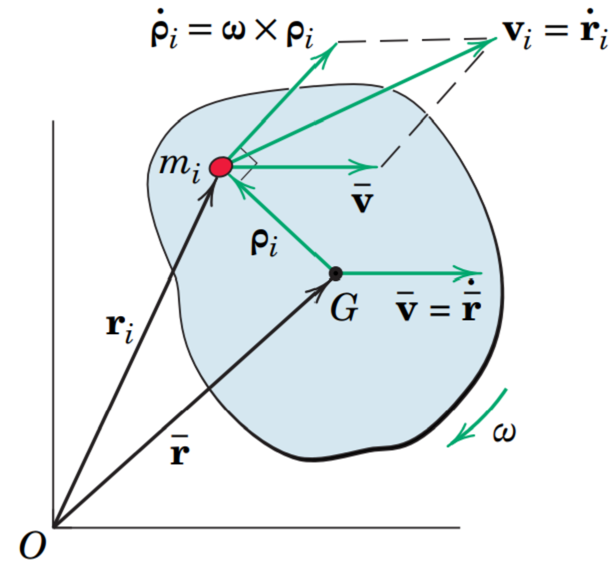
6/8 Impulse-Momentum Equations

Angular Momentum

$$H_G = \bar{I}\omega$$

$$\Sigma M_G = \dot{H}_G$$

$$(H_G)_1 + \int_{t_1}^{t_2} \Sigma M_G \, dt = (H_G)_2$$



6/8 Impulse-Momentum Equations

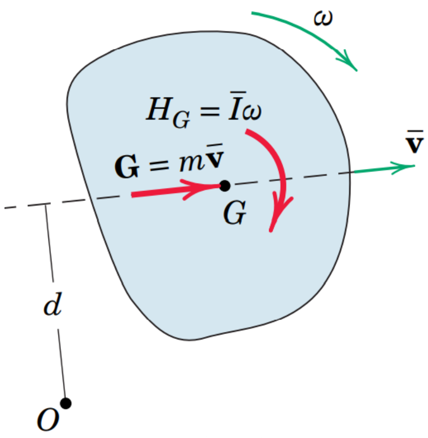
Angular Momentum

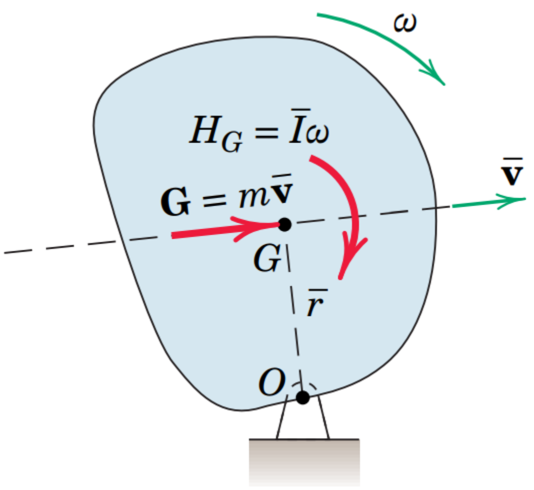
$$H_O = \bar{I}\omega + m\bar{v}d$$

$$H_O = I_O\omega$$

$$\Sigma M_O = \dot{H}_O$$

$$(H_O)_1 + \int_{t_1}^{t_2} \Sigma M_O \, dt = (H_O)_2$$





6/8 Impulse-Momentum Equations

Conservation of Momentum

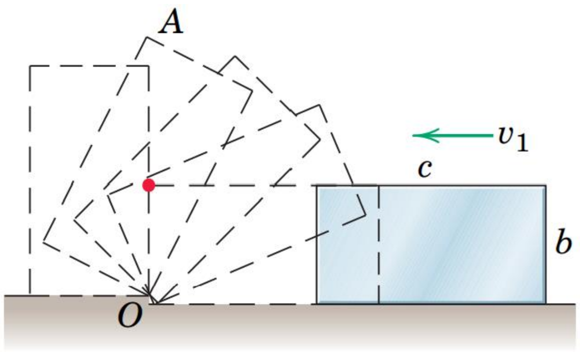
$$\mathbf{G}_1 = \mathbf{G}_2$$

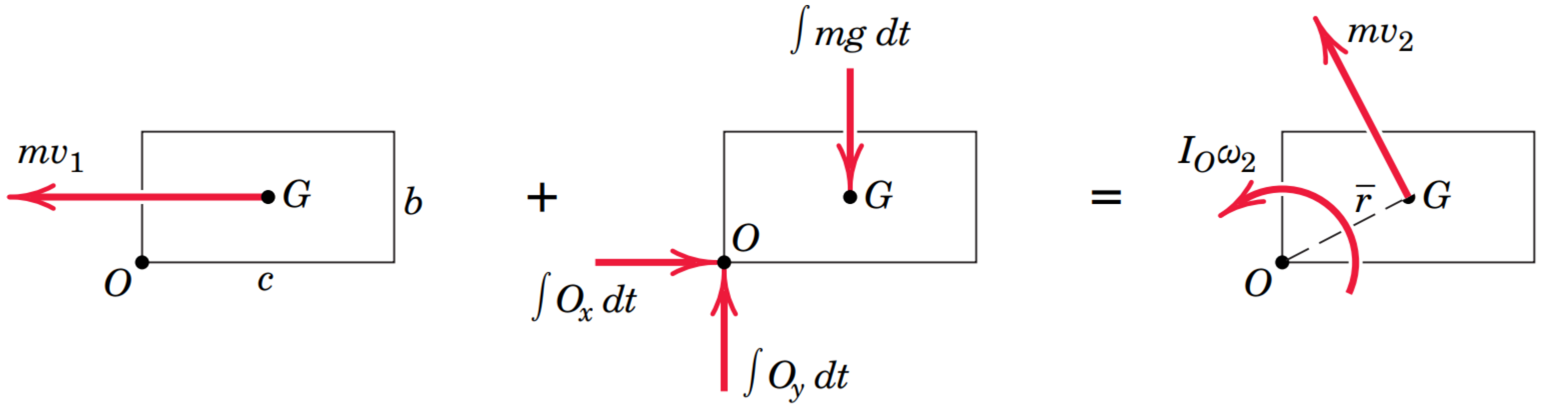
$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$$

$$(\mathbf{H}_G)_1 = (\mathbf{H}_G)_2$$

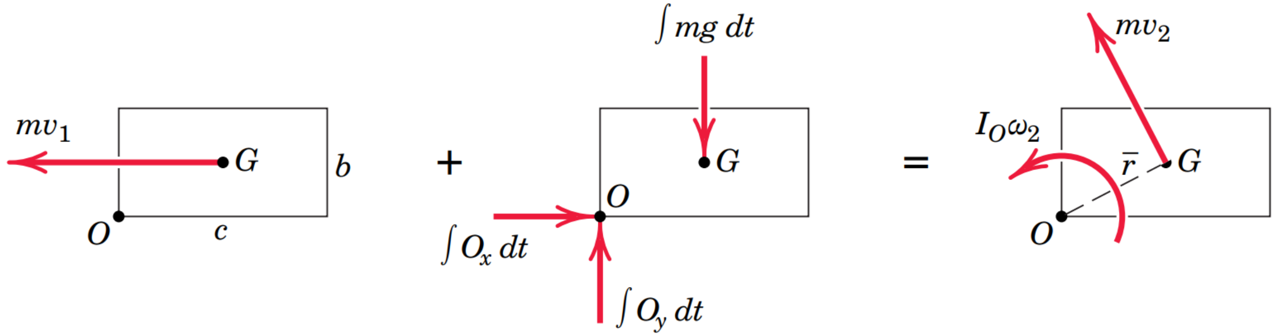
SAMPLE PROBLEM 6/16

The uniform rectangular block of dimensions shown is sliding to the left on the horizontal surface with a velocity v_1 when it strikes the small step at O. Assume negligible rebound at the step and compute the minimum value of v_1 which will permit the block to pivot freely about O and just reach the standing position A with no velocity. Compute the percentage energy loss n for $b = c$.





Impact of Rigid Bodies

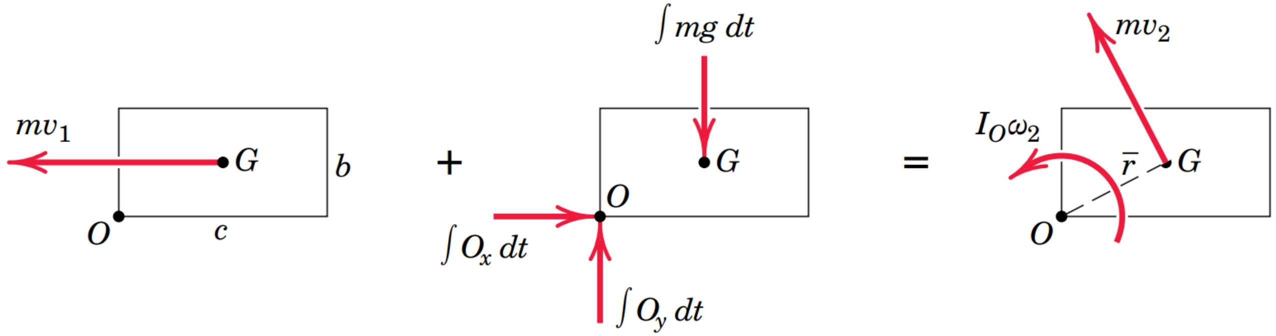


I. Collision

$$[H_O = I_O\omega] \qquad (H_O)_2 = \left\{\frac{1}{12}m(b^2+c^2) + m\left[\left(\frac{c}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right]\right\}\omega_2$$

$$= \frac{m}{3}(b^2+c^2)\omega_2$$

$$[(H_O)_1 = (H_O)_2] \qquad mv_1\frac{b}{2} = \frac{m}{3}(b^2+c^2)\omega_2 \qquad \omega_2 = \frac{3v_1 b}{2(b^2+c^2)}$$



II. Rotation about *O*

$$[T_2 + V_2 = T_3 + V_3] \qquad \frac{1}{2}I_O{\omega_2}^2 + 0 = 0 + mg\left[\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2} - \frac{b}{2}\right]$$

$$\frac{1}{2}\frac{m}{3}(b^2 + c^2)\left[\frac{3v_1 b}{2(b^2 + c^2)}\right]^2 = \frac{mg}{2}(\sqrt{b^2 + c^2} - b)$$

$$v_1 = 2\sqrt{\frac{g}{3}\left(1 + \frac{c^2}{b^2}\right)(\sqrt{b^2 + c^2} - b)}$$

$$n = \frac{|\Delta E|}{E} = \frac{\frac{1}{2}m{v_1}^2 - \frac{1}{2}I_O{\omega_2}^2}{\frac{1}{2}m{v_1}^2} = 1 - \frac{k_O^2{\omega_2}^2}{{v_1}^2} = 1 - \left(\frac{b^2 + c^2}{3}\right)\left[\frac{3b}{2(b^2 + c^2)}\right]^2$$

$$= 1 - \frac{3}{4\left(1 + \frac{c^2}{b^2}\right)} \qquad\qquad n = 62.5\% \text{ for } b = c$$