Semnan University
Faculty of Mechanical Engineering

دانشگاه سمنان

دانشکده مهندسی مکانیک

دانشکده مهندسی مکانیک

درس مبانی برق ۱

نیمسال اول ۹۸-۹۹

Electrical Engineering

Principles and Applications

Allan R. Hambley

5th Edition

❑ *Contents:*

INTRODUCTION

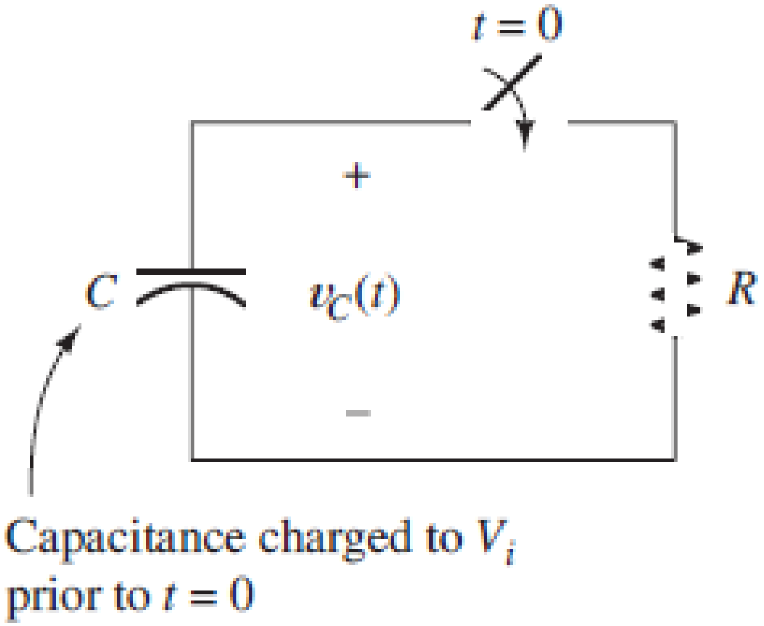
- Transient and steady-state response concepts
- Solve first-order *RC* or *RL* circuits
- First-order circuits time constant.
- Solve *RLC* circuits in dc steady-state conditions
- Solve second-order circuits
- Second-order system natural frequency and damping ratio

4.1 FIRST-ORDER RC CIRCUITS

- Discharge of a Capacitance through a Resistance
  - Current equation at the top node:

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0$$

$$\Longrightarrow \quad RC\frac{dv_C(t)}{dt} + v_C(t) = 0$$



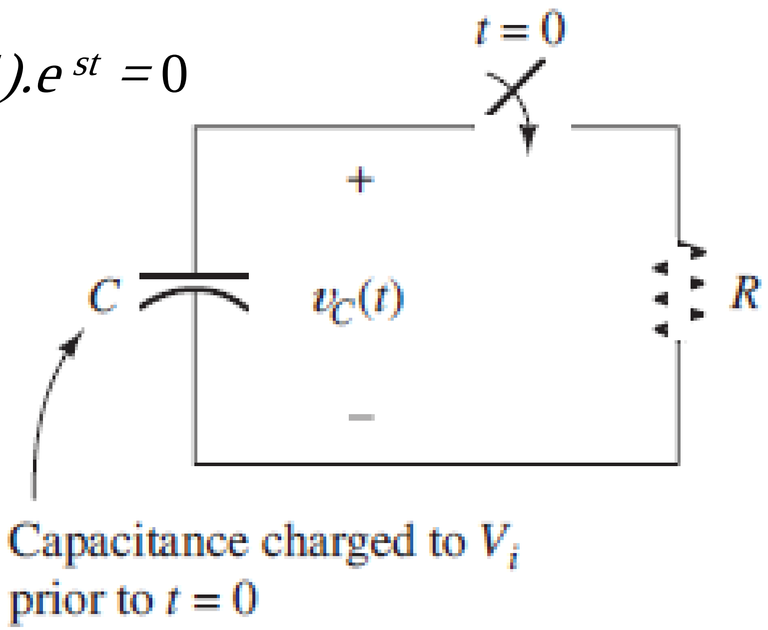
$$v_C(t) = Ke^{st}$$

$$\Longrightarrow \quad RCKse^{st} + Ke^{st} = 0$$

4.1 FIRST-ORDER RC CIRCUITS

- Discharge of a Capacitance through a Resistance

$$RCKse^{st} + Ke^{st} = 0 \Rightarrow K(RCs+1).e^{st} = 0$$



- Solve for "s":

$$s = \frac{-1}{RC} \Rightarrow v_C(t) = Ke^{-t/RC}$$

- Initial condition for "K":

$$v_C(0+) = V_i$$

$$\Rightarrow v_C(0+) = V_i = Ke^0 = K$$

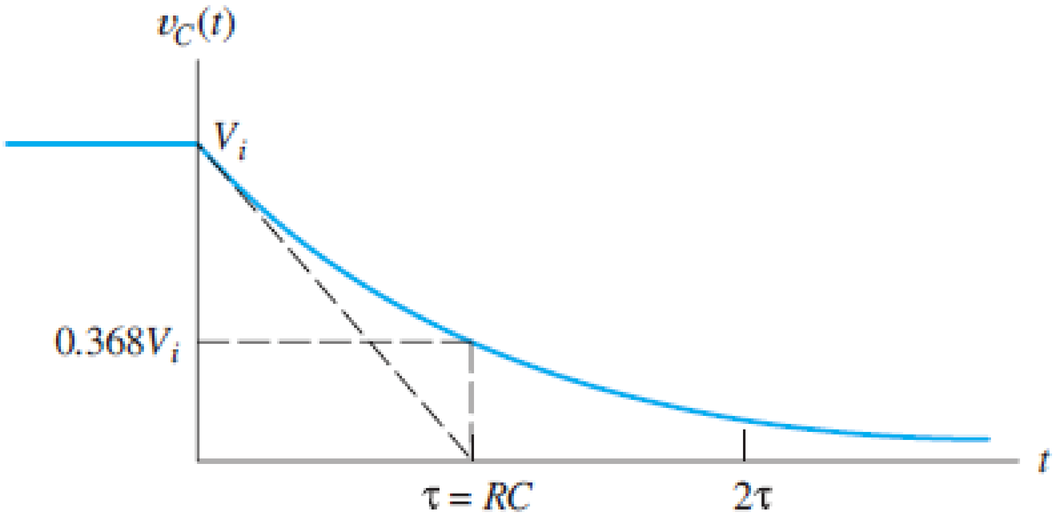
$$\Rightarrow v_C(t) = V_i e^{-t/RC}$$

4.1 FIRST-ORDER RC CIRCUITS

- Time constant

$$v_C(t) = V_i e^{-t/RC}$$

$$\tau = RC$$



- Voltage decays to 36.8%
- Applying *RC* circuits in timing applications

4.1 FIRST-ORDER RC CIRCUITS

- Charging a Capacitance from a DC Source through a Resistance

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t) - V_s}{R} = 0$$

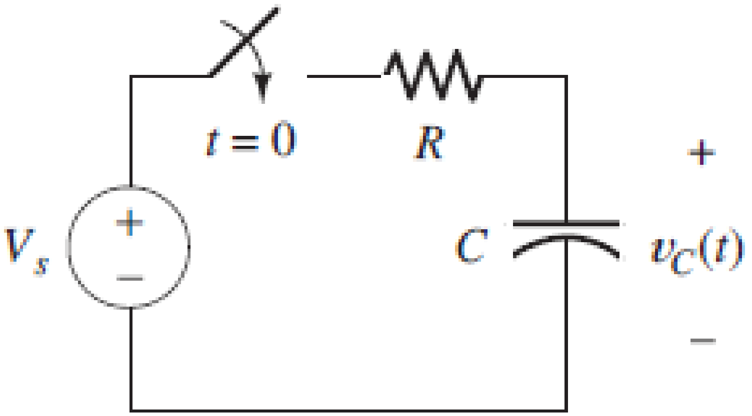
$$\Rightarrow RC\frac{dv_C(t)}{dt} + v_C(t) = V_s$$

$$v_C(0+) = v_C(0-) = 0$$

$$v_C(t) = K_1 + K_2 e^{st}$$

$$(1 + RCs)K_2 e^{st} + K_1 = V_s$$

$$\Rightarrow K_1 = V_s \qquad s = \frac{-1}{RC} \qquad \Rightarrow v_C(t) = V_s + K_2 e^{-t/RC}$$



4.1 FIRST-ORDER RC CIRCUITS

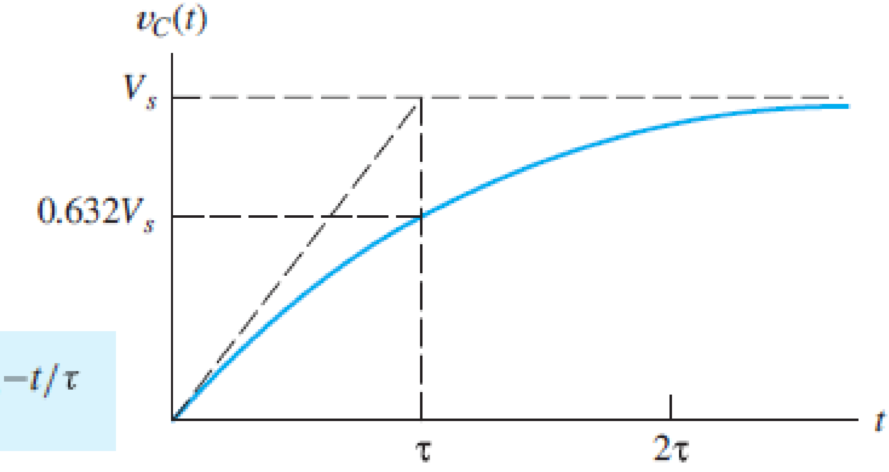
- Charging a Capacitance from a DC Source through a Resistance

$$v_C(0+) = 0 = V_s + K_2 e^0 = V_s + K_2$$

$$\Rightarrow K_2 = -V_s$$

$$\Rightarrow v_C(t) = V_s - V_s e^{-t/RC}$$

$$\tau = RC \Rightarrow v_C(t) = V_s - V_s e^{-t/\tau}$$



- Steady-state (forced) and Transient response

4.2 DC STEADY STATE

- Constant dc sources:
  - Capacitance:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

    - ✓ For steady-state conditions with dc sources, **capacitances** behave as **open circuits**
  - Inductance:

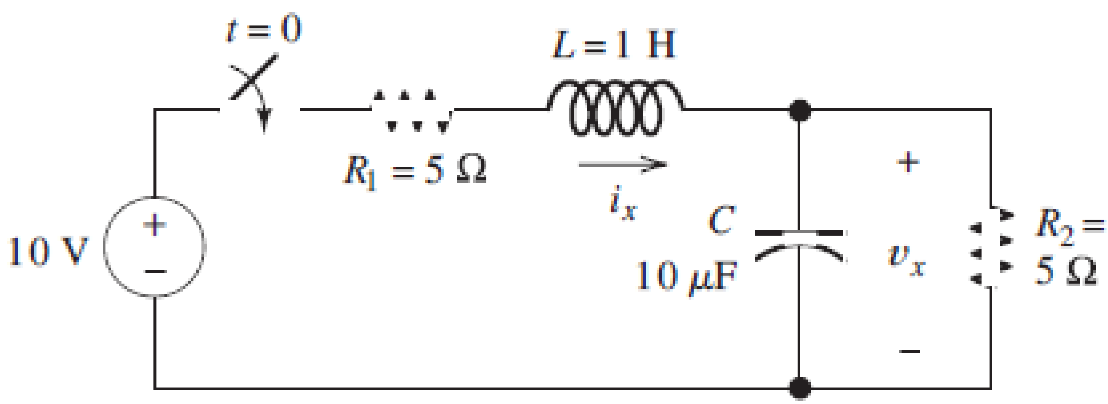
$$v_L(t) = L\frac{di_L(t)}{dt}$$

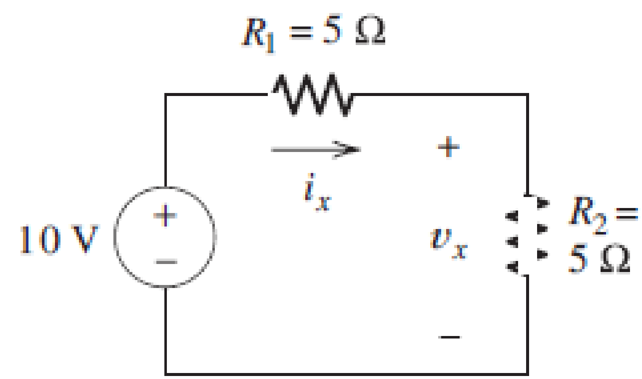
    - ✓ For steady-state conditions with dc sources, **inductances** behave as **short circuits**

- Determining the forced (steady-state) response for *RLC* circuits with dc sources:
  - 1. Replace capacitances with open circuits.
  - 2. Replace inductances with short circuits.
  - 3. Solve the remaining circuit.

4.2 DC STEADY STATE

Example 4.1 Steady-State DC Analysis



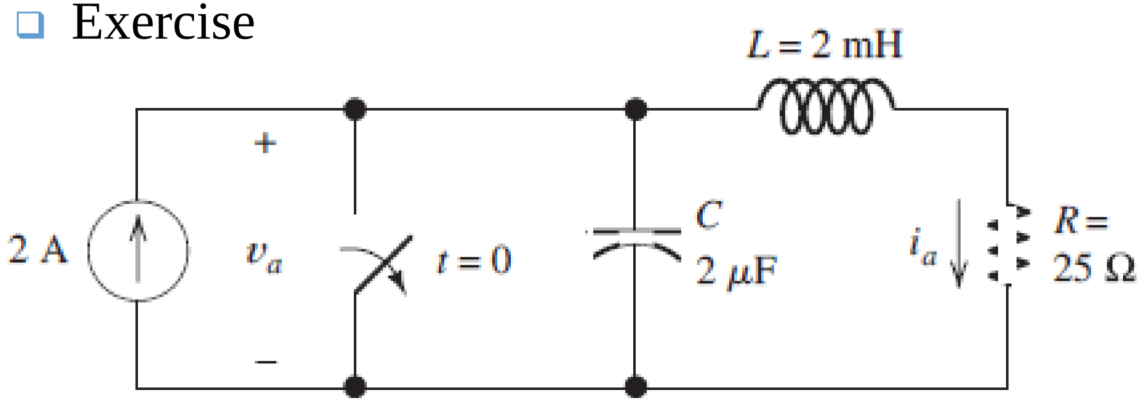


$$i_x = \frac{10}{R_1 + R_2} = 1\ \text{A}$$

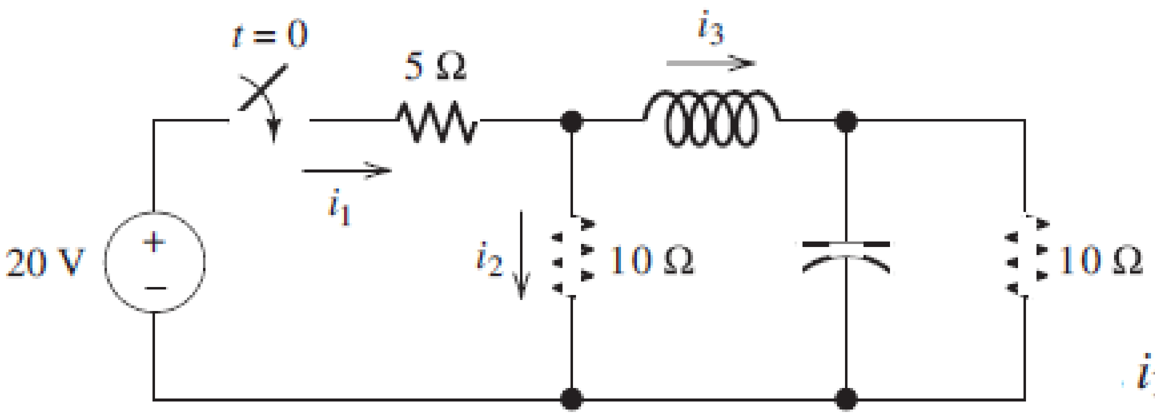
$$v_x = R_2 i_x = 5\ \text{V}$$

4.2 DC STEADY STATE

- Exercise



$v_a = 50$ V, $i_a = 2$ A;

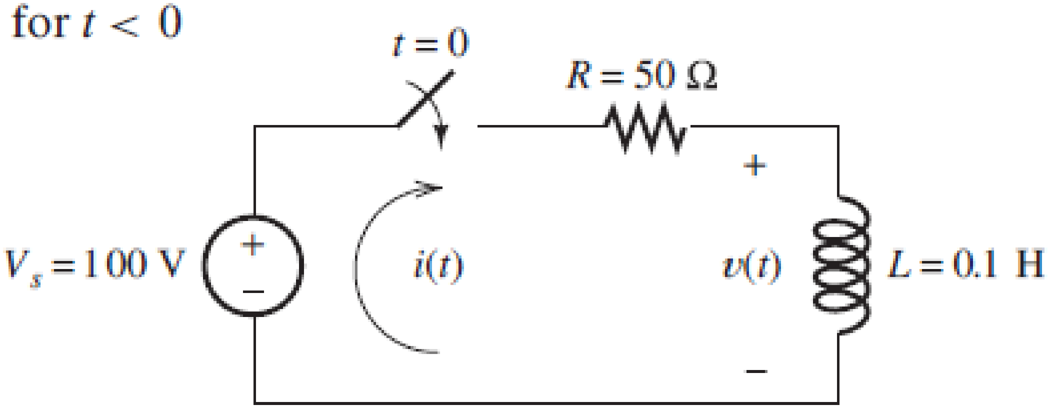


$i_1 = 2$ A, $i_2 = 1$ A, $i_3 = 1$ A.

4.3 RL CIRCUITS

Example 4.2 *RL* Transient Analysis

- For current: $i(t) = 0 \quad \text{for } t < 0$



- KVL equation:

$$Ri(t) + L\frac{di}{dt} = V_s$$

$$i(t) = K_1 + K_2 e^{st}$$

$$\Longrightarrow RK_1 + (RK_2 + sLK_2)e^{st} = V_s$$

$$\Longrightarrow \quad K_1 = \frac{V_s}{R} = 2, \quad s = \frac{-R}{L} \quad \Longrightarrow \quad i(t) = 2 + K_2 e^{-tR/L}$$

4.3 RL CIRCUITS

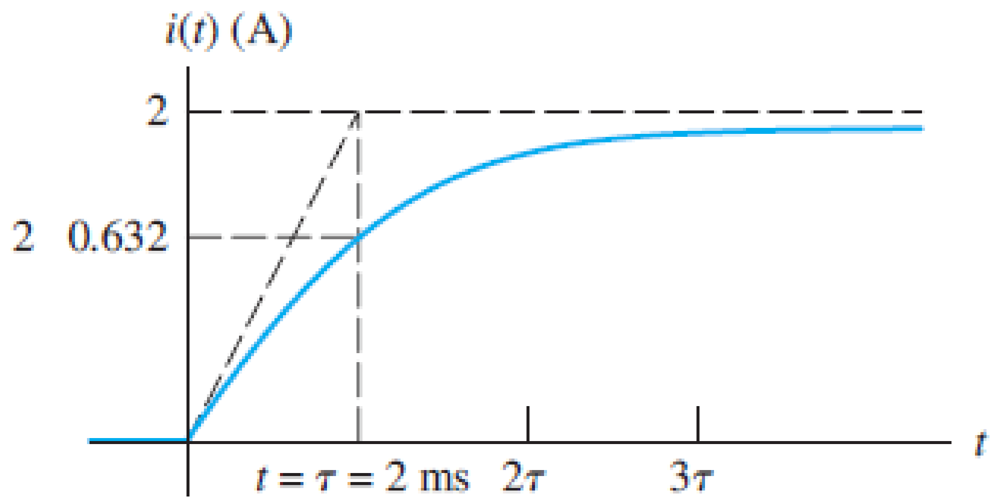
Example 4.2 *RL* Transient Analysis

$$i(t) = 2 + K_2 e^{-tR/L}$$

$$i(0+) = 0 = 2 + K_2 e^0 = 2 + K_2 \implies K_2 = -2.$$

$$\implies i(t) = 2 - 2e^{-t/\tau} \quad \text{for } t > 0$$

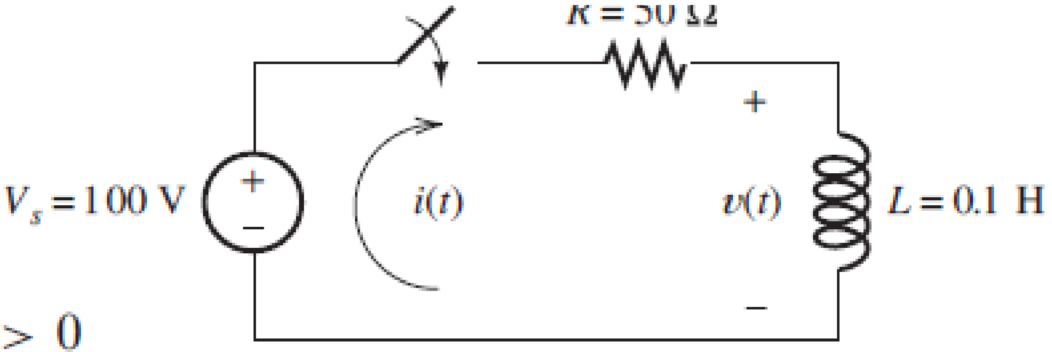
$$\tau = \frac{L}{R}$$



4.3 RL CIRCUITS

Example 4.2 *RL* Transient Analysis

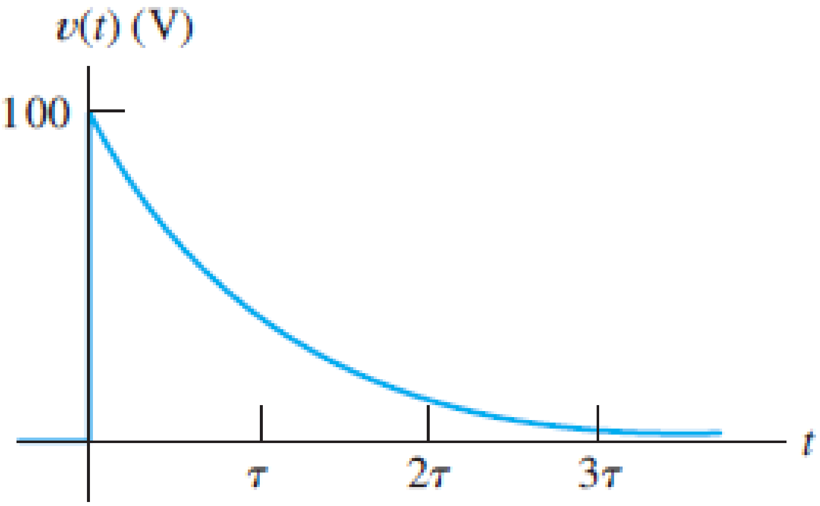
- For voltage:



$$v(t) = 0 \qquad \text{for } t < 0$$

$$v(t) = 100 - 50i(t) \qquad \text{for } t > 0$$

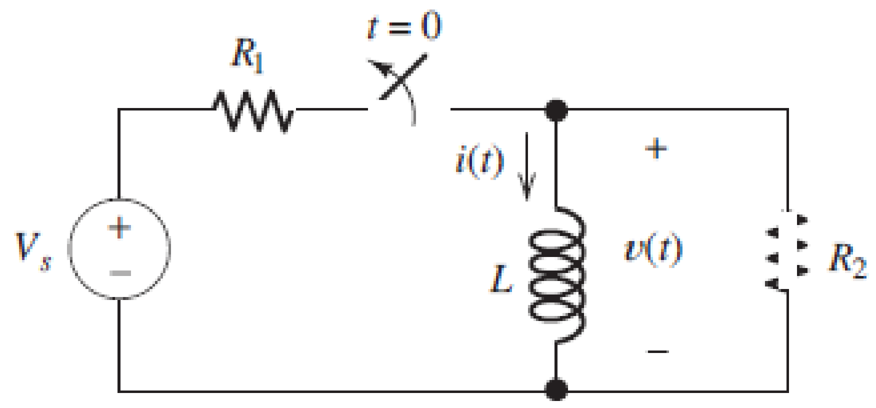
$$\Longrightarrow \quad v(t) = 100e^{-t/\tau}$$



4.3 RL CIRCUITS

Example 4.3 *RL* Transient Analysis

- Prior to $t = 0$:
  - the inductor behaves as a short circuit



$$v(t) = 0 \qquad \text{for } t < 0$$

$$i(t) = \frac{V_s}{R_1} \qquad \text{for } t < 0$$

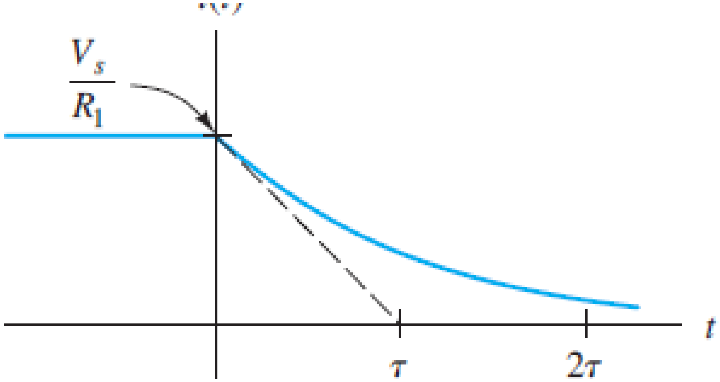
$$i(t) = Ke^{-t/\tau} \qquad \text{for } t > 0 \qquad \tau = \frac{L}{R_2}$$

$$i(0+) = \frac{V_s}{R_1} = Ke^{-0} = K$$

4.3 RL CIRCUITS

Example 4.3 *RL* Transient Analysis

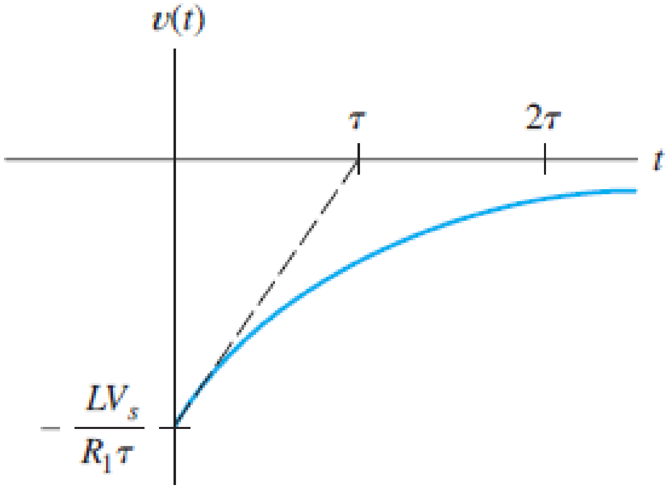
$$i(t) = \frac{V_s}{R_1} e^{-t/\tau} \qquad \text{for } t > 0$$



$$v(t) = L\frac{di(t)}{dt}$$

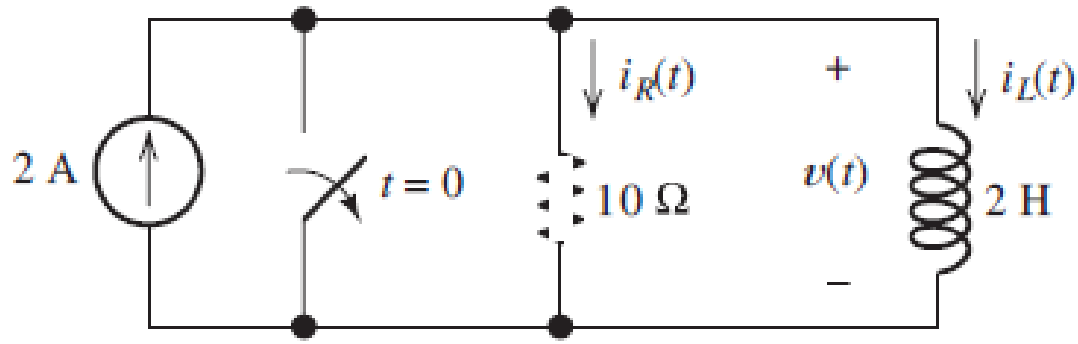
$$= 0 \qquad \text{for } t < 0$$

$$= -\frac{LV_s}{R_1\tau} e^{-t/\tau} \qquad \text{for } t > 0$$



4.3 RL CIRCUITS

- Exercise



$$\frac{v(t)}{R}+\frac{1}{L}\int_0^t v(x)dx+0=2$$

$$\Rightarrow \frac{dv(t)}{dt}+\frac{R}{L}v(t)=0$$

$$\Rightarrow v(t)=K\exp(-t/\tau)$$

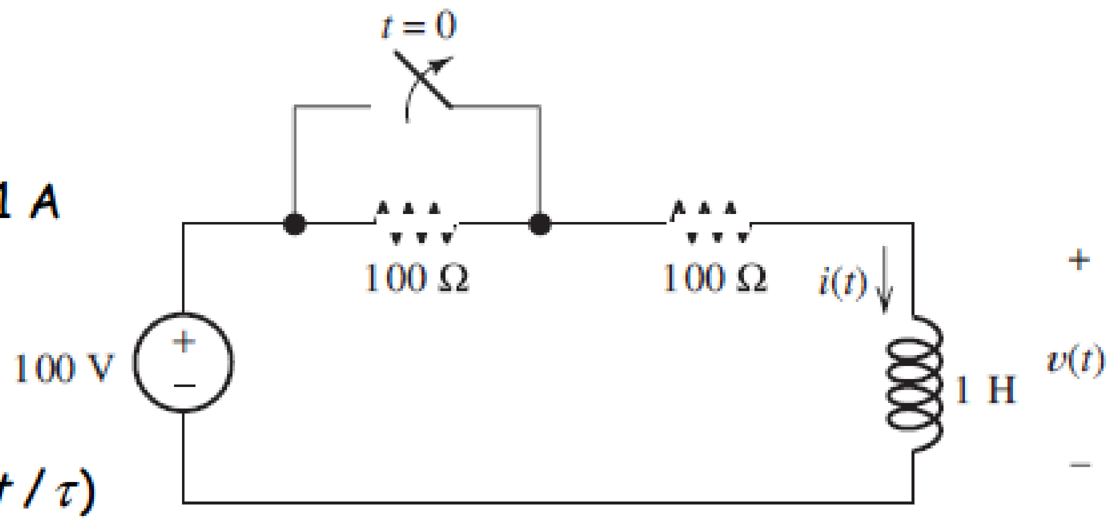
$$\tau=L/R=0.2\,s$$

$$v(0+)=20=K \Rightarrow v(t)=20\exp(-t/\tau) \quad i_R=v/R=2\exp(-t/\tau)$$

$$\Rightarrow i_L(t)=\frac{1}{L}\int_0^t v(x)dx=\frac{1}{2}\left[-20\tau\exp(-x/\tau)\right]_0^t=2-2\exp(-t/\tau)$$

4.3 RL CIRCUITS

- Exercise



$i(0-) = 1\ A \Rightarrow i(0+) = 1\ A$

$$\frac{di(t)}{dt} + 200i(t) = 100$$

$$\Rightarrow i(t) = K_1 + K_2 \exp(-t/\tau)$$

$$\tau = 1/200 = 5\,\text{ms}$$

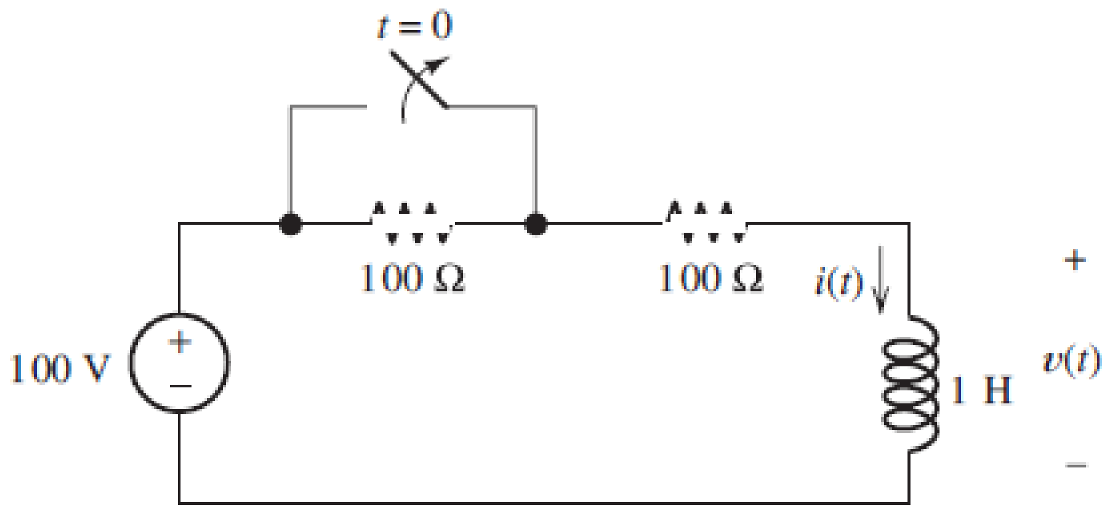
$$i(\infty) = K_1 = 100/200 = 0.5\ A.$$

$$i(0+) = 1 = K_1 + K_2 \Rightarrow K_2 = 0.5 \Rightarrow i(t) = 1.0\ A \text{ for } t < 0$$

$$= 0.5 + 0.5\exp(-t/\tau) \text{ for } t > 0$$

4.3 RL CIRCUITS

- Exercise

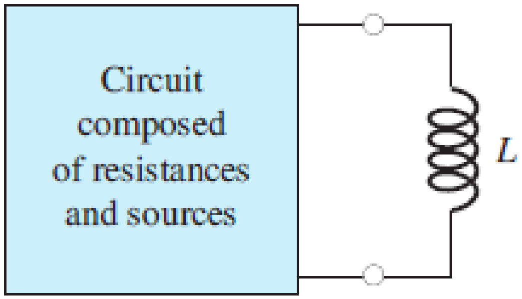


$$i(t) = 1.0 \ \ A \text{ for } t < 0$$
$$= 0.5 + 0.5\exp(-t/\tau) \text{ for } t > 0$$

$$\Longrightarrow v(t) = L\frac{di(t)}{dt}$$
$$= 0 \ \ V \text{ for } t < 0$$
$$= -100\exp(-t/\tau) \text{ for } t > 0.$$

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

- Circuits that contain one energy-storage element
  - ✓ (either an inductance or a capacitance)



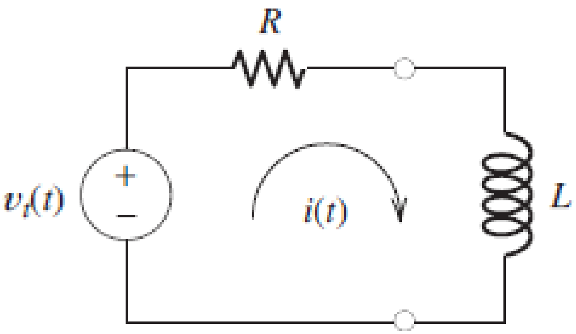
- KVL:

$$L\frac{di(t)}{dt} + Ri(t) = v_t(t)$$

$$\Longrightarrow \frac{L}{R}\frac{di(t)}{dt} + i(t) = \frac{v_t(t)}{R}$$

$$\Longrightarrow \tau\frac{dx(t)}{dt} + x(t) = f(t)$$

forcing function



- ✓ Linear first-order differential equation

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

- Solution of the Differential Equation

$$\tau \frac{dx(t)}{dt} + x(t) = f(t)$$

- General solution: *X(t) = Xc(t) + Xp(t)*
  - ✓ **Complementary solution** *Xc(t)* (homogeneous equation, natural response)

$$\tau \frac{dx_c(t)}{dt} + x_c(t) = 0$$

  - ✓ **Particular solution** *Xp(t)* (forced response)
    - Satisfies the differential equation
    - May not be consistent with the initial conditions

$$\tau \frac{dx_p(t)}{dt} + x_p(t) = f(t)$$

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

- Example $\tau \frac{dx(t)}{dt} + x(t) = f(t)$ $\quad f(t) = 10\cos(200t)$
  - Forced response:

$$\tau \frac{dx_p(t)}{dt} + x_p(t) = f(t) \Longrightarrow x_p(t) = A\cos(200t) + B\sin(200t)$$

  - Natural response:

$$\tau \frac{dx_c(t)}{dt} + x_c(t) = 0 \Longrightarrow \frac{dx_c(t)/dt}{x_c(t)} = \frac{-1}{\tau}$$

$$\Longrightarrow \ln[x_c(t)] = \frac{-t}{\tau} + c \Longrightarrow x_c(t) = e^{(-t/\tau + c)} = e^c e^{-t/\tau}$$

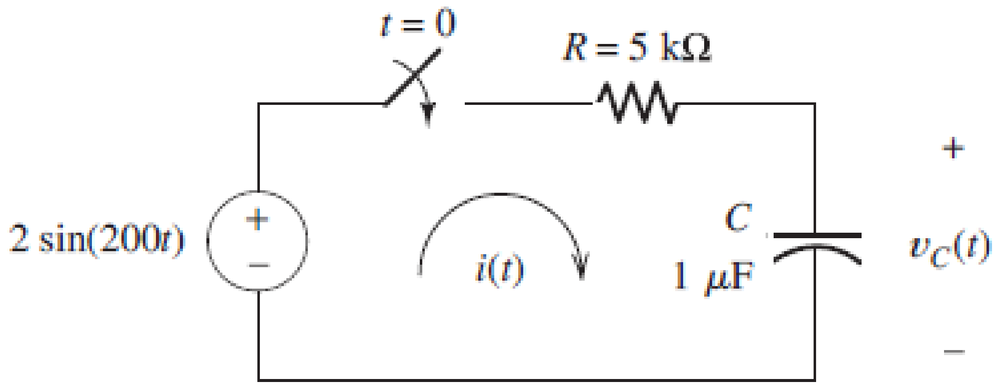
$$\Longrightarrow x_c(t) = Ke^{-t/\tau}$$

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

- Step-by-Step Solution:
  1) Write the circuit differential equation
  2) Find a particular solution (forced response)
  3) Find the complementary solution (natural response)
  4) Add particular and complementary solutions
  5) Use initial conditions to find parameters

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

Example 4.4 Transient Analysis of an *RC* Circuit with a Sinusoidal Source



KVL:

$v_C(0) = 1\ \mathrm{V}$

$$Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_C(0) - 2\sin(200t) = 0 \Longrightarrow R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 400\cos(200t)$$

$$\Longrightarrow RC\frac{di(t)}{dt} + i(t) = 400\,C\cos(200t)$$

$$\Longrightarrow 5\times 10^{-3}\frac{di(t)}{dt} + i(t) = 400\times 10^{-6}\cos(200t)$$

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

Example 4.4 Transient Analysis of an *RC* Circuit with a Sinusoidal Source

Find a particular solution:

Guessing at the form of $i_p(t)$, possibly including some unknown constants

$$RC\frac{di(t)}{dt} + i(t) = 400C\cos(200t) \Longrightarrow i_p(t) = A\cos(200t) + B\sin(200t)$$

$$\Longrightarrow -A\sin(200t) + B\cos(200t) + A\cos(200t) + B\sin(200t) = 400\times 10^{-6}\cos(200t)$$

Equating the coefficients:

$$\begin{aligned} -A + B &= 0 \\ B + A &= 400\times 10^{-6} \end{aligned} \Longrightarrow \begin{aligned} A &= 200\times 10^{-6} = 200\ \mu\text{A} \\ B &= 200\times 10^{-6} = 200\ \mu\text{A} \end{aligned}$$

$$\Longrightarrow i_p(t) = 200\cos(200t) + 200\sin(200t)\ \mu\text{A}$$

$$\Longrightarrow i_p(t) = 200\sqrt{2}\cos(200t - 45^\circ)$$

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

Example 4.4 Transient Analysis of an *RC* Circuit with a Sinusoidal Source

Obtain homogenous solution:

$$RC\frac{di(t)}{dt} + i(t) = 0 \quad \Longrightarrow \quad i_c(t) = Ke^{-t/RC} = Ke^{-t/\tau}$$

General solution: $i(t) = 200\cos(200t) + 200\sin(200t) + Ke^{-t/RC}\ \mu\text{A}$

Determine value K by using initial condition

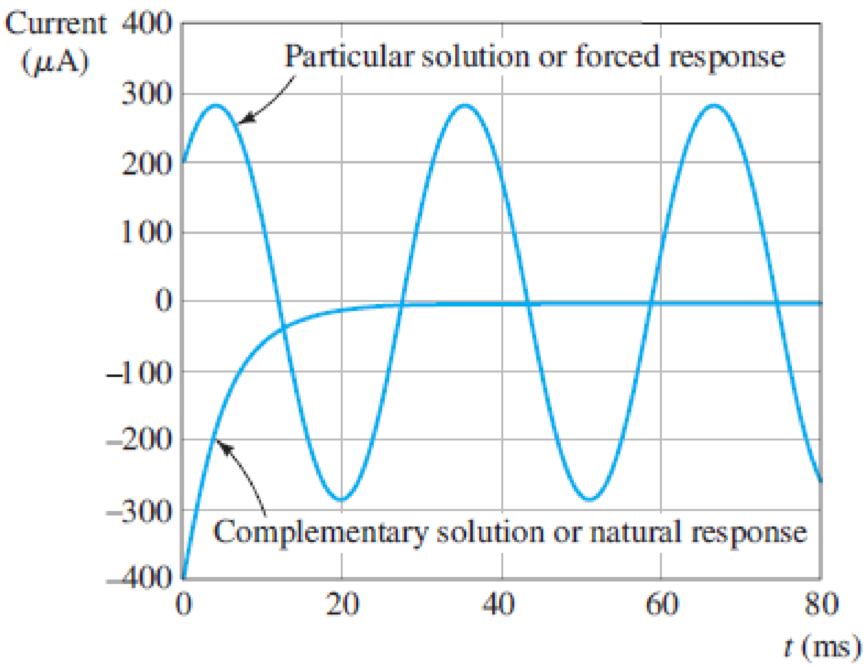
$$v_C(0+) = 1 \qquad v_R(0+) = -1\text{ V.} \quad \Longrightarrow \quad i(0+) = \frac{v_R(0+)}{R} = \frac{-1}{5000} = -200\ \mu\text{A}$$

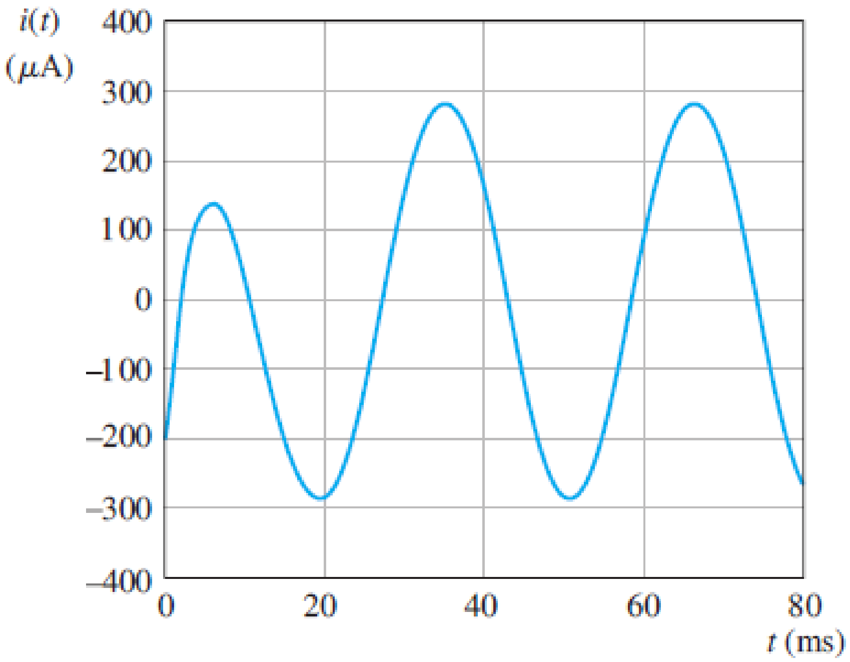
$$\Longrightarrow \quad i(0+) = -200 = 200 + K\ \mu\text{A} \quad \Longrightarrow \quad K = -400\ \mu\text{A}$$

$$\Longrightarrow \quad i(t) = 200\cos(200t) + 200\sin(200t) - 400e^{-t/RC}\ \mu\text{A}$$

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

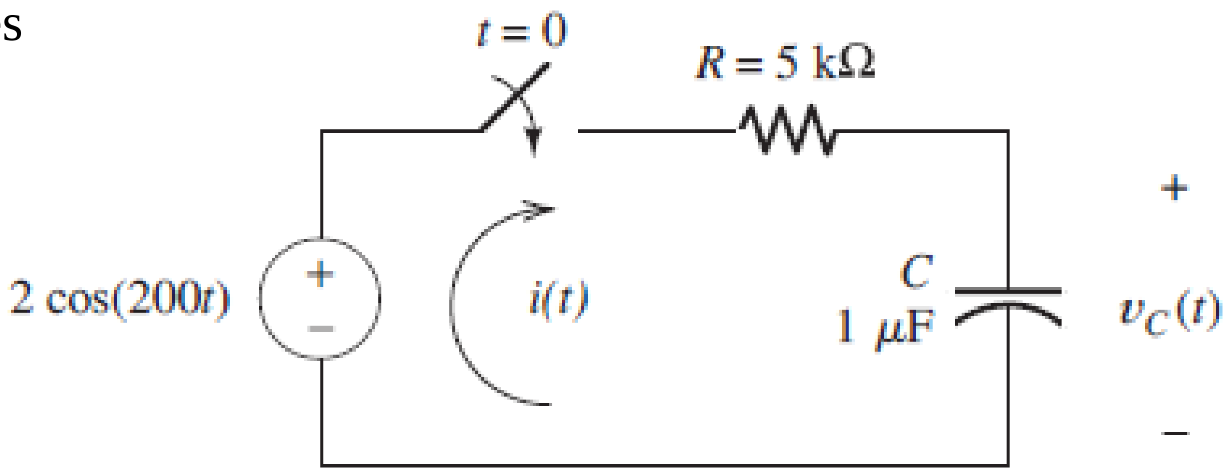
Example 4.4 Transient Analysis of an *RC* Circuit with a Sinusoidal Source





4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

- Exercises



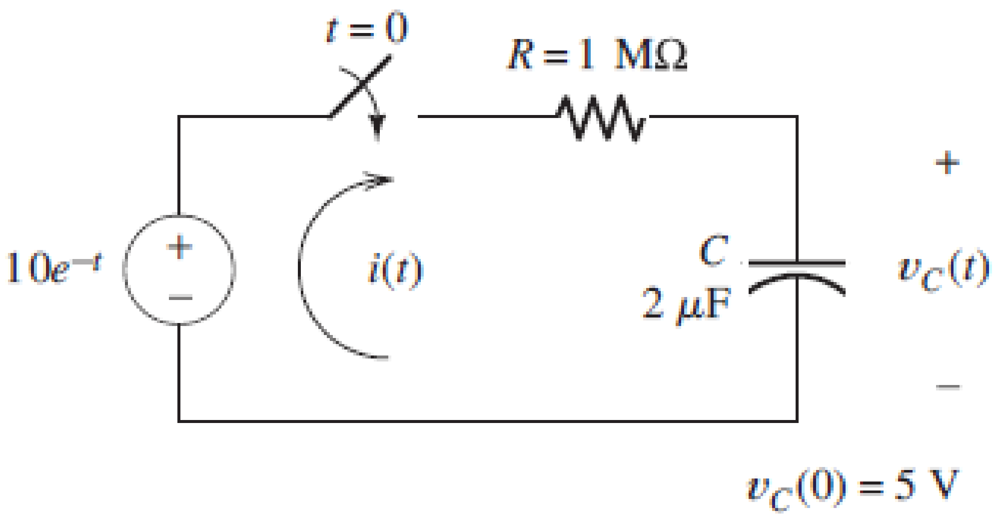
$v_C(0) = 0$

$$i(t) = -200\sin(200t) + 200\cos(200t) + 200e^{-t/RC}\ \mu\text{A}.$$

$$\tau = RC = 5\text{ ms}.$$

4.4 RC AND RL CIRCUITS WITH GENERAL SOURCES

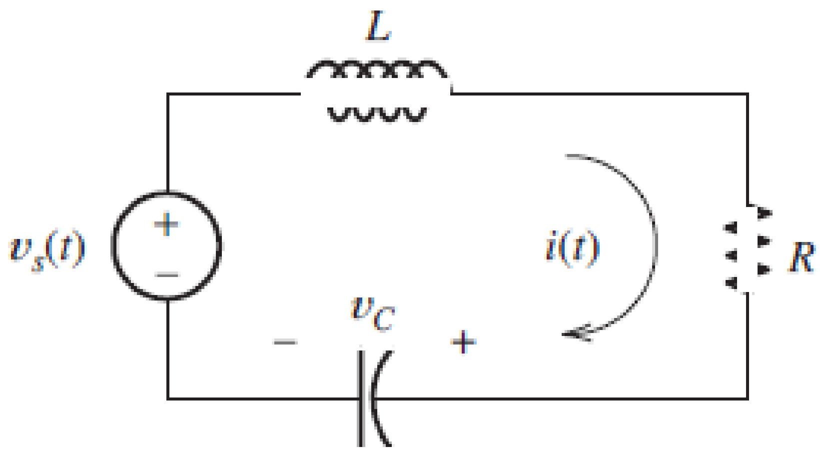
- Exercises



$$i(t) = 20e^{-t} - 15e^{-t/2}\ \mu\text{A}.$$

4.5 SECOND-ORDER CIRCUITS

- Contain two energy-storage elements
  - ✓ One inductance and one capacitance
  - ✓ Either in series or in parallel



- Differential Equation

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)dt + v_C(0) = v_s(t)$$

$$\Longrightarrow L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv_s(t)}{dt}$$

$$\Longrightarrow \frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$

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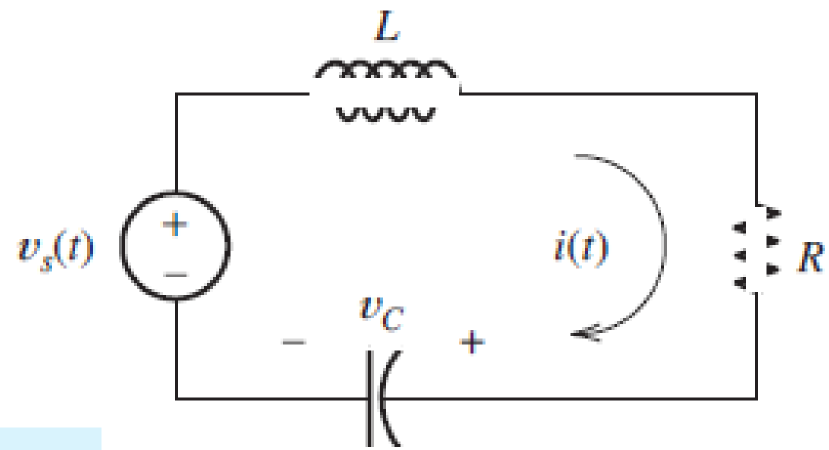
4.5 SECOND-ORDER CIRCUITS

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$



- Damping coefficient: $\alpha = \frac{R}{2L}$
- Undamped resonant frequency: $\omega_0 = \frac{1}{\sqrt{LC}}$
- Forcing function: $f(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$

$$\Rightarrow \quad \frac{d^2i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + \omega_0^2 i(t) = f(t)$$

4.5 SECOND-ORDER CIRCUITS

- Solution of the Second-Order Equation

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t)$$

- General solution:
  - ✓ Particular solution (*Xp(t)*) plus complementary solution (*Xc(t)*)

$$x(t) = x_p(t) + x_c(t)$$

- Particular solution

$$\frac{d^2x_p(t)}{dt^2} + 2\alpha\frac{dx_p(t)}{dt} + \omega_0^2 x_p(t) = f(t)$$

- Complementary solution

$$\frac{d^2x_c(t)}{dt^2} + 2\alpha\frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0$$

4.5 SECOND-ORDER CIRCUITS

- Complementary solution:

$$\frac{d^2x_c(t)}{dt^2} + 2\alpha\frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0$$

$$x_c(t) = Ke^{st}.$$

$$\Longrightarrow \quad s^2Ke^{st} + 2\alpha sKe^{st} + \omega_0^2Ke^{st} = 0$$

$$\Longrightarrow \quad (s^2 + 2\alpha s + \omega_0^2)Ke^{st} = 0 \qquad \Longrightarrow \quad s^2 + 2\alpha s + \omega_0^2 = 0$$

- Characteristic equation:
  - ✓ Damping ratio

$$\zeta = \frac{\alpha}{\omega_0}$$

- Roots of the characteristic equation:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

4.5 SECOND-ORDER CIRCUITS

- 3 Cases:
  1) Overdamped case $(\zeta > 1)$
     - Roots are real and distinct

$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

  2) Critically damped case
     - Roots are real and equal

$$x_c(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$

  3) Underdamped case
     - Roots are complex

$$s_1 = -\alpha + j\omega_n$$
$$s_2 = -\alpha - j\omega_n$$

$$j = \sqrt{-1}$$

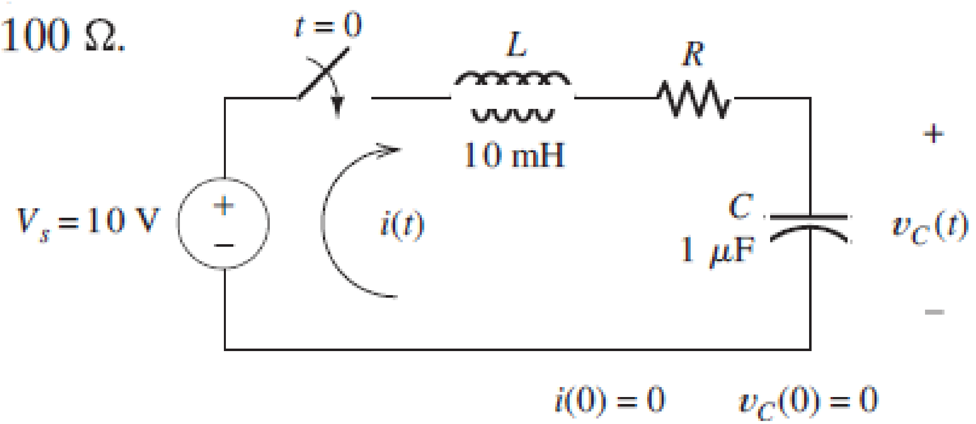
$$\omega_n = \sqrt{\omega_0^2 - \alpha^2}$$ **natural frequency**

$$\Rightarrow \quad x_c(t) = K_1 e^{-\alpha t}\cos(\omega_n t) + K_2 e^{-\alpha t}\sin(\omega_n t)$$

4.5 SECOND-ORDER CIRCUITS

Example 4.5 Analysis of a Second-Order Circuit with a DC Source

Solve for $v_C(t)$ if $R = 300, 200,$ and $100\ \Omega$.



$$i(t) = C\frac{dv_C(t)}{dt}$$

$$L\frac{di(t)}{dt} + Ri(t) + v_C(t) = V_s$$

$$\Longrightarrow LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = V_s$$

$$\Longrightarrow \frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_s}{LC}$$

4.5 SECOND-ORDER CIRCUITS

Example 4.5 Analysis of a Second-Order Circuit with a DC Source

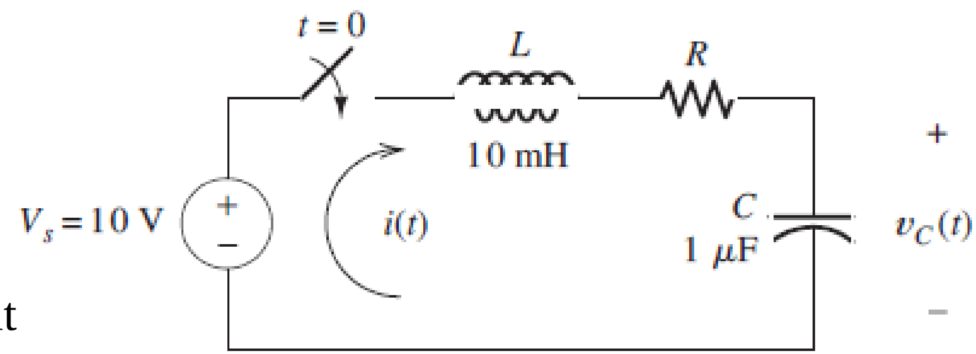
- Only dc source:

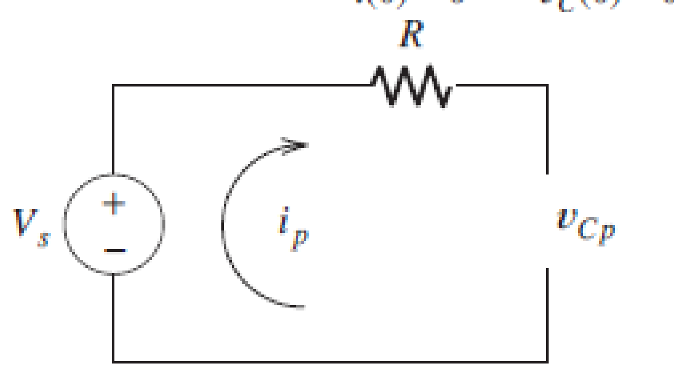
  Replacing:

  - Inductance by a short circuit
  - Capacitance by an open circuit

  Leads to particular solution:

$$\Rightarrow \quad v_{Cp}(t) = V_s = 10 \text{ V}$$





4.5 SECOND-ORDER CIRCUITS

Example 4.5 Analysis of a Second-Order Circuit with a DC Source

Homogenous solution:

Case I ($R = 300\ \Omega$)

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^4$$

$$\alpha = \frac{R}{2L} = 1.5 \times 10^4$$

$$\zeta = \alpha/\omega_0 = 1.5 \qquad \zeta > 1 \text{ overdamped}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$= -1.5 \times 10^4 - \sqrt{(1.5 \times 10^4)^2 - (10^4)^2}$$

$$= -2.618 \times 10^4$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$= -0.3820 \times 10^4$$

$$\Longrightarrow v_C(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \qquad \Longrightarrow 10 + K_1 + K_2 = 0$$

4.5 SECOND-ORDER CIRCUITS

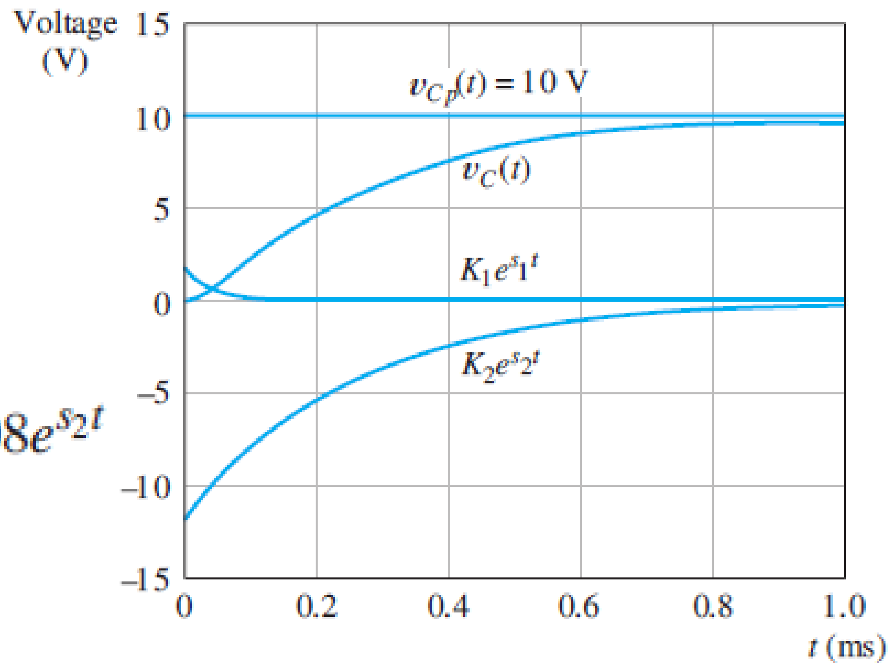
Example 4.5 Analysis of a Second-Order Circuit with a DC Source

$$i(0) = 0 \Rightarrow i(t) = C\frac{dv_C(t)}{dt} \Rightarrow \frac{dv_C(0)}{dt} = 0$$

$$\Rightarrow s_1K_1 + s_2K_2 = 0$$

$$\Rightarrow \begin{aligned} K_1 &= 1.708 \\ K_2 &= -11.708 \end{aligned}$$

$$\Rightarrow v_C(t) = 10 + 1.708e^{s_1 t} - 11.708e^{s_2 t}$$



4.5 SECOND-ORDER CIRCUITS

Example 4.5 Analysis of a Second-Order Circuit with a DC Source

Case II ($R = 200\ \Omega$)

$$\alpha = \frac{R}{2L} = 10^4$$

$$\zeta = \alpha/\omega_0 = 1$$ critically damped case.

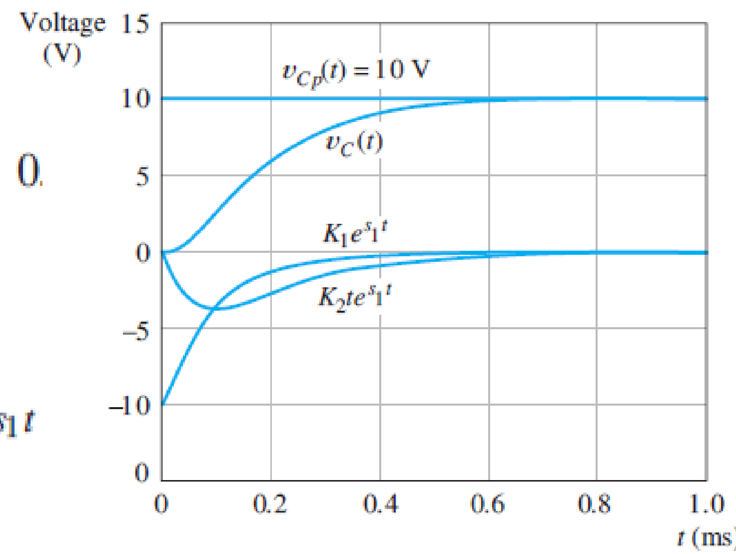
$$\Rightarrow s_1 = s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha = -10^4$$

$$\Rightarrow v_C(t) = 10 + K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$

initial conditions $v_C(0) = 0 \quad dv_C(0)/dt = 0$

$$\Rightarrow \begin{matrix} 10 + K_1 = 0 \\ s_1 K_1 + K_2 = 0 \end{matrix} \quad \Rightarrow \begin{matrix} K_1 = -10 \\ K_2 = -10^5 \end{matrix}$$

$$\Rightarrow v_C(t) = 10 - 10e^{s_1 t} - 10^5 t e^{s_1 t}$$



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4.5 SECOND-ORDER CIRCUITS

Example 4.5 Analysis of a Second-Order Circuit with a DC Source

Case III ($R = 100\ \Omega$) $\alpha = \frac{R}{2L} = 5000$ $\zeta = \alpha/\omega_0 = 0.5$ underdamped

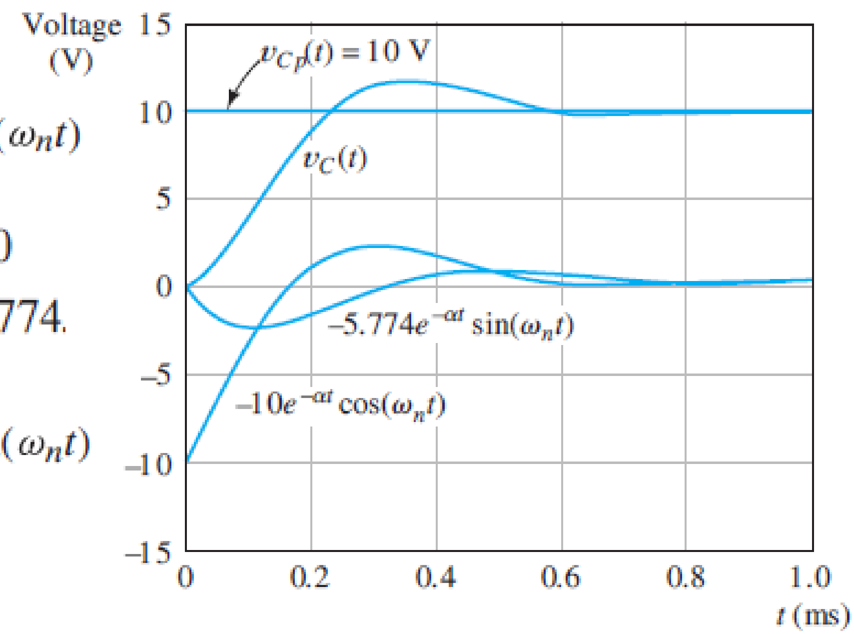
$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} = 8660$$

$$v_C(t) = 10 + K_1 e^{-\alpha t}\cos(\omega_n t) + K_2 e^{-\alpha t}\sin(\omega_n t)$$

$$10 + K_1 = 0$$
$$-\alpha K_1 + \omega_n K_2 = 0$$

$$K_1 = -10$$
$$K_2 = -5.774$$

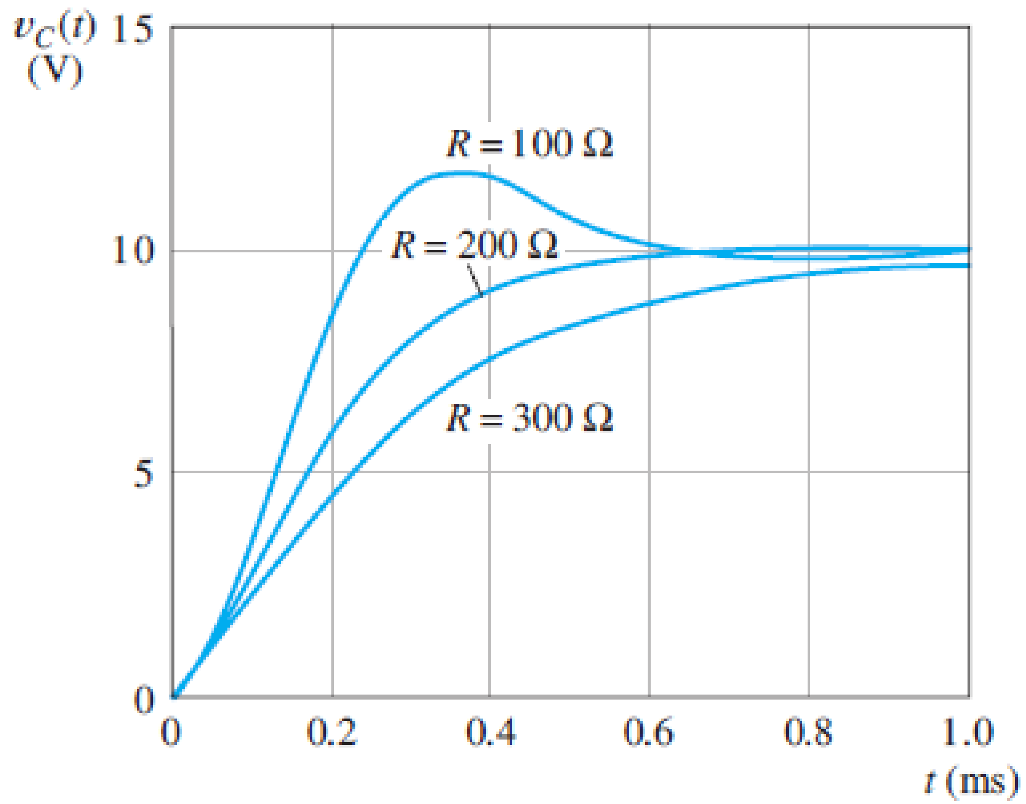
$$v_C(t) = 10 - 10e^{-\alpha t}\cos(\omega_n t) - 5.774e^{-\alpha t}\sin(\omega_n t)$$



4.5 SECOND-ORDER CIRCUITS

Example 4.5 Analysis of a Second-Order Circuit with a DC Source

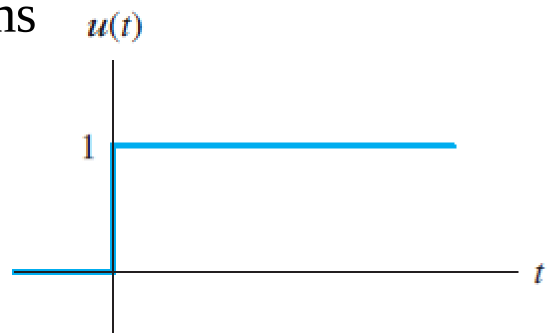
- Comparison:



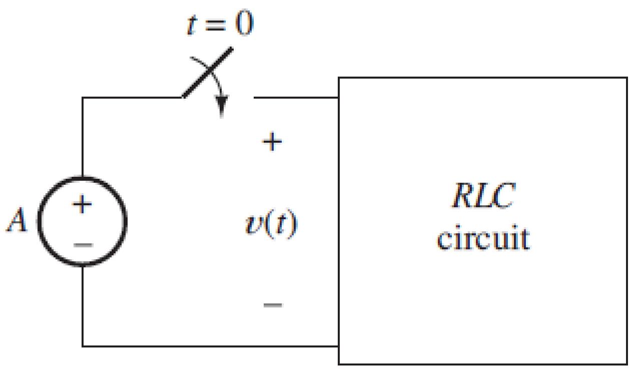
4.5 SECOND-ORDER CIRCUITS

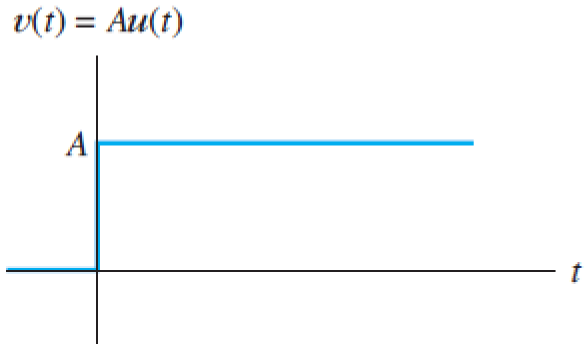
- Normalized Step Response of Second-Order Systems
  - Step function

$$u(t) = 0 \quad t < 0$$
$$= 1 \quad t \geq 0$$



$$v(t) = Au(t)$$





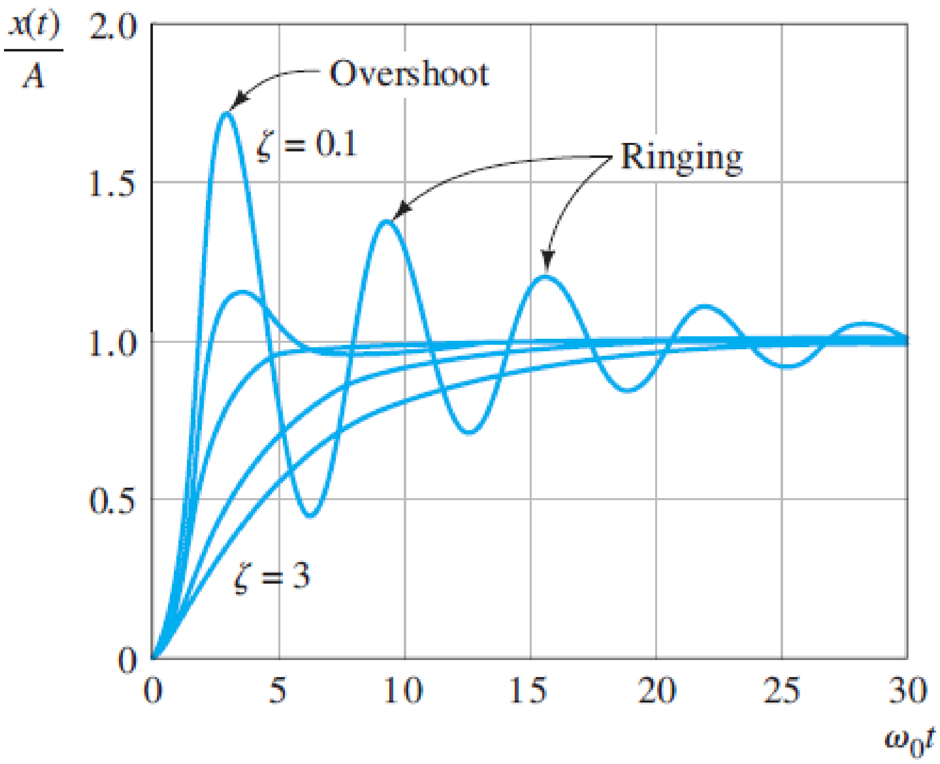
4.5 SECOND-ORDER CIRCUITS

- Normalized Step Response of Second-Order Systems

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = Au(t)$$

undamped resonant frequency ω_0

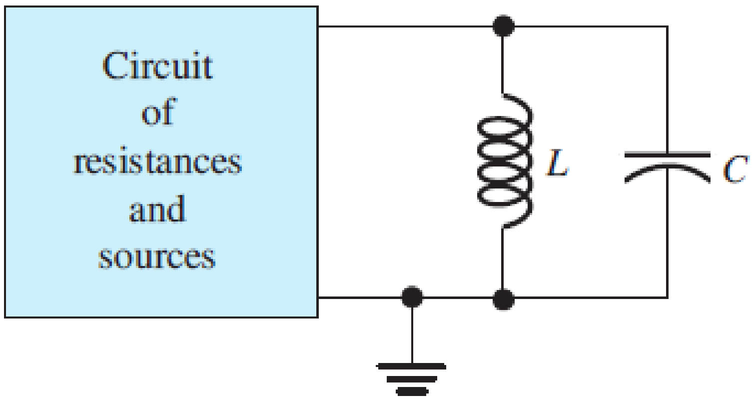
damping ratio $\zeta = \alpha/\omega_0$.

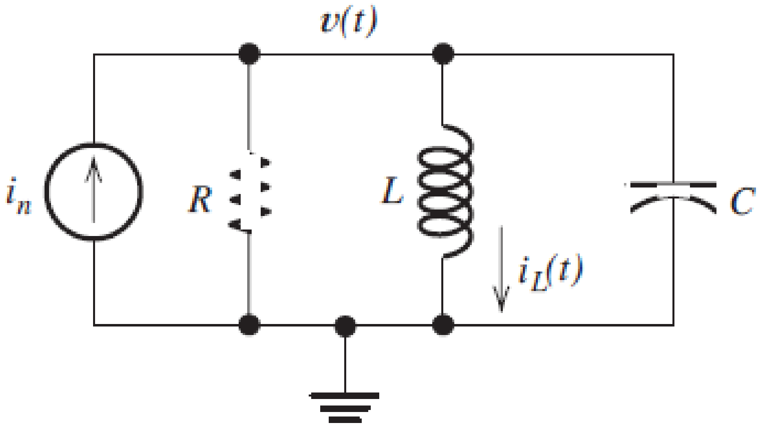


- Overshoot
- Ringing

4.5 SECOND-ORDER CIRCUITS

- Circuits with Parallel *L* and *C*





- Writing a KCL equation at the top node:

$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) + \frac{1}{L}\int_0^t v(t)\,dt + i_L(0) = i_n(t)$$

4.5 SECOND-ORDER CIRCUITS

- Simplification

$$C\frac{dv(t)}{dt} + \frac{1}{R}v(t) + \frac{1}{L}\int_0^t v(t)\,dt + i_L(0) = i_n(t)$$

$$\Longrightarrow \quad C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = \frac{di_n(t)}{dt}$$

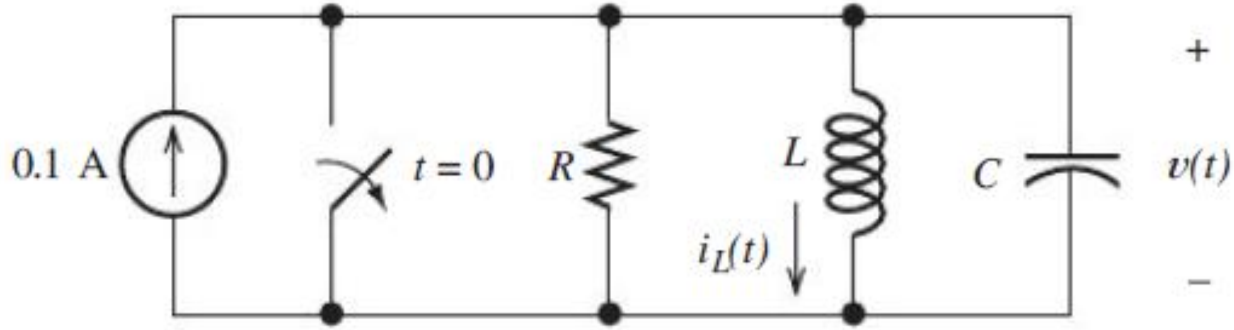
$$\Longrightarrow \quad \frac{d^2v(t)}{dt^2} + \frac{1}{RC}\frac{dv(t)}{dt} + \frac{1}{LC}v(t) = \frac{1}{C}\frac{di_n(t)}{dt}$$

- ✓ Damping coefficient $\alpha = \frac{1}{2RC}$
- ✓ Undamped resonant frequency $\omega_0 = \frac{1}{\sqrt{LC}}$
- ✓ Forcing function $f(t) = \frac{1}{C}\frac{di_n(t)}{dt}$

$$\Longrightarrow \quad \frac{d^2v(t)}{dt^2} + 2\alpha\frac{dv(t)}{dt} + \omega_0^2 v(t) = f(t)$$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.9



$R = 25\ \Omega. \quad L = 1\text{ mH} \quad C = 0.1\ \mu\text{F}$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 10^{-7}}} = 10^5 \qquad \alpha = \frac{1}{2RC} = 2 \times 10^5 \qquad \zeta = \frac{\alpha}{\omega_0} = 2$$

At $t = 0+$, the KCL equation $\quad 0.1 = \dfrac{v(0+)}{R} + i_L(0+) + Cv'(0+)$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.9

$$v(0+) = v(0-) = 0$$
$$i_L(0+) = i_L(0-) = 0 \quad \Longrightarrow \quad v'(0+) = 10^6 \ \ V/s$$

particular solution $\Longrightarrow$ steady-state conditions $\Longrightarrow$ $v_p(t) = 0$

homogeneous solution $\Longrightarrow$ circuit is overdamped $(\zeta > 1)$

$$\Longrightarrow \begin{aligned} s_1 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -373.2 \times 10^3 \\ s_2 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -26.79 \times 10^3 \end{aligned}$$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.9

$$\Longrightarrow \quad v(t) = K_1 \exp(s_1 t) + K_2 \exp(s_2 t)$$

initial conditions

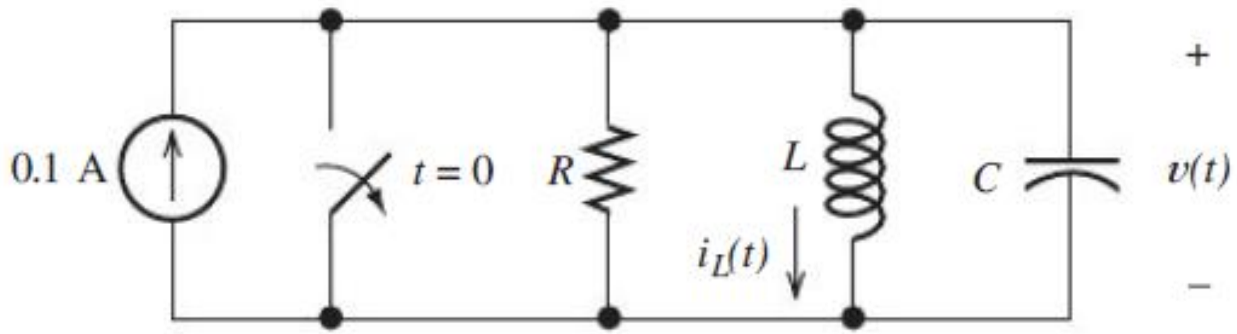
$$v(0+) = 0 = K_1 + K_2 \qquad v'(0+) = 10^6 = K_1 s_1 + K_2 s_2$$

$$\Longrightarrow \quad K_1 = -2.887 \text{ and } K_2 = 2.887$$

$$\Longrightarrow \quad v(t) = 2.887[\exp(s_2 t) - \exp(s_1 t)]$$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.10



$R = 50\ \Omega \quad L = 1\ \text{mH} \quad C = 0.1\ \mu\text{F}$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 10^{-7}}} = 10^5 \quad \alpha = \frac{1}{2RC} = 10^5 \quad \zeta = \frac{\alpha}{\omega_0} = 1$$

At $t = 0+$, the KCL equation $\quad 0.1 = \frac{v(0+)}{R} + i_L(0+) + Cv'(0+)$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.10

$$v(0+) = v(0-) = 0$$
$$i_L(0+) = i_L(0-) = 0$$
$$\Longrightarrow v'(0+) = 10^6 \ \ V/s$$

particular solution $\Longrightarrow$ steady-state conditions $\Longrightarrow$ $v_p(t) = 0$

homogeneous solution

$$\Longrightarrow s_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -10^5 \quad s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -10^5$$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.10

$$\Longrightarrow \quad v(t) = K_1 \exp(s_1 t) + K_2 t \exp(s_1 t)$$

initial conditions

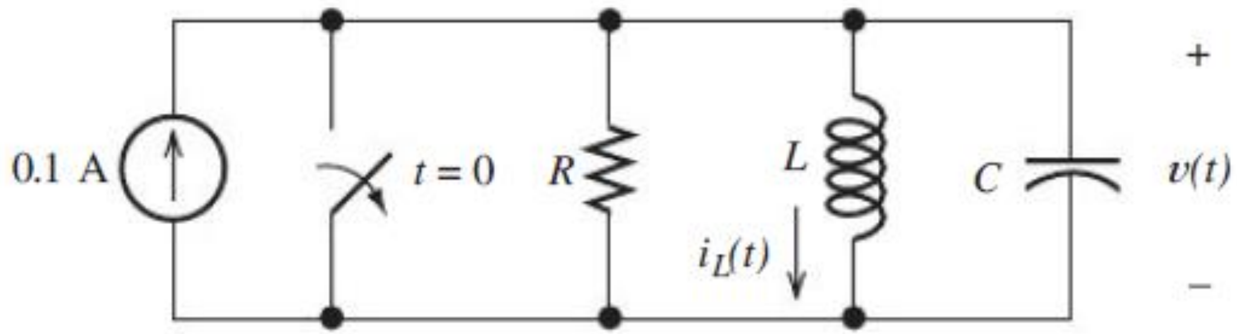
$$v(0+) = 0 = K_1 \qquad v'(0+) = 10^6 = K_1 s_1 + K_2$$

$$\Longrightarrow \quad K_2 = 10^6$$

$$\Longrightarrow \quad v(t) = 10^6 t \exp(-10^5 t)$$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.11



$R = 250\ \Omega \quad L = 1\ \text{mH} \quad C = 0.1\ \mu\text{F}$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 10^{-7}}} = 10^5 \qquad \alpha = \frac{1}{2RC} = 2 \times 10^4 \qquad \zeta = \frac{\alpha}{\omega_0} = 0.2$$

At $t = 0+$, the KCL equation

$$0.1 = \frac{v(0+)}{R} + i_L(0+) + Cv'(0+)$$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.11

$$v(0+) = v(0-) = 0$$
$$i_L(0+) = i_L(0-) = 0$$
$$\Longrightarrow v'(0+) = 10^6 \ V/s$$

particular solution $\Longrightarrow$ steady-state conditions $\Longrightarrow$ $v_p(t) = 0$

homogeneous solution

$$\Longrightarrow \omega_n = \sqrt{\omega_0^2 - \alpha^2} = 97.98 \times 10^3$$

4.5 SECOND-ORDER CIRCUITS

- Exercise 4.11

$$\Rightarrow \quad v(t) = K_1 \exp(-\alpha t)\cos(\omega_n t) + K_2 \exp(-\alpha t)\sin(\omega_n t)$$

initial conditions

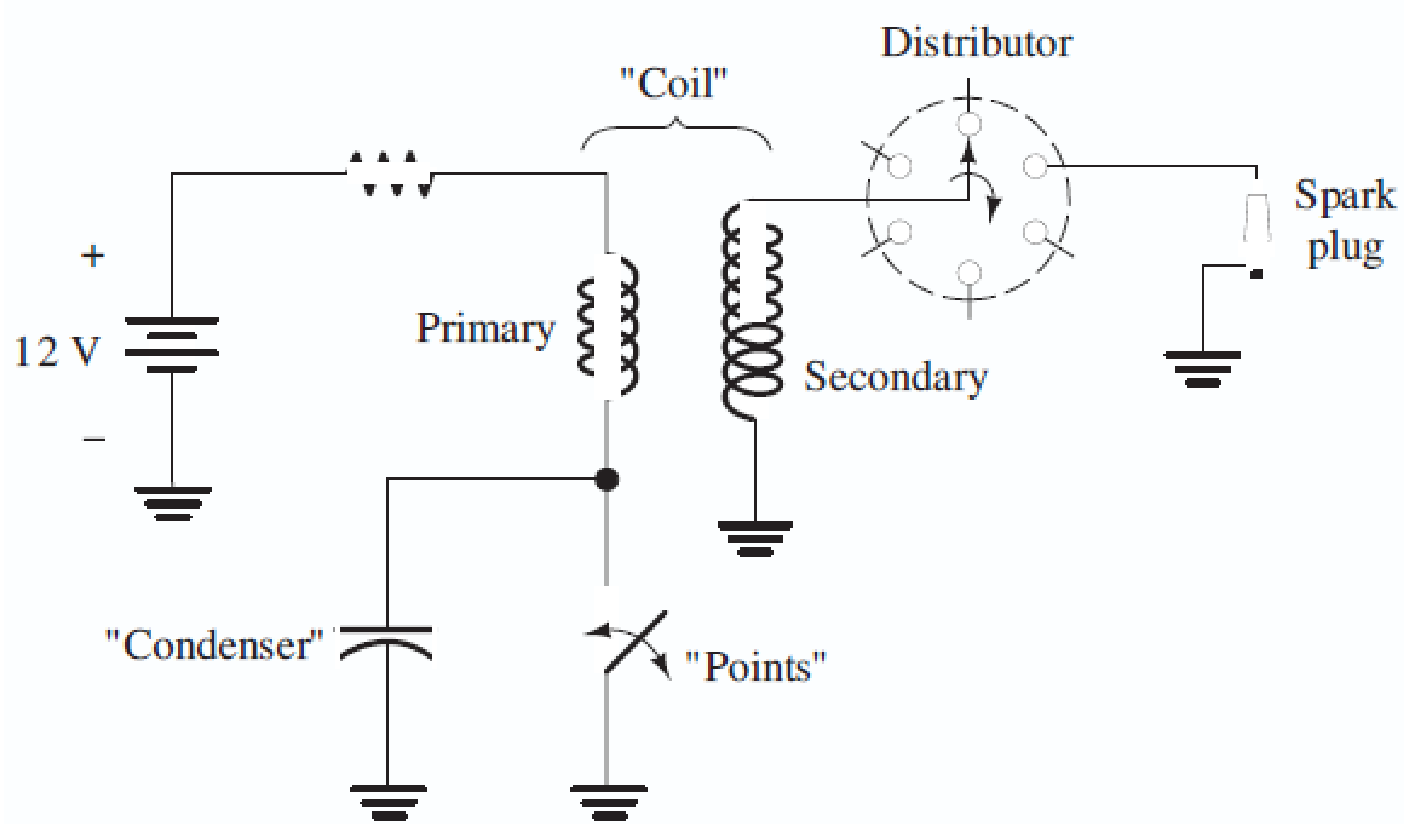
$$v(0+) = 0 = K_1 \qquad v'(0+) = 10^6 = -\alpha K_1 + \omega_n K_2$$

$$\Rightarrow \quad K_2 = 10.21$$

$$\Rightarrow \quad v(t) = 10.21 \exp(-2\times 10^4 t)\sin(97.98\times 10^3 t)\ \text{V}$$

4.5 SECOND-ORDER CIRCUITS

- Classic ignition for an internal-combustion engine



EXERCISES

- P4.4
- P4.5
- P4.21
- P4.22
- P4.23
- P4.33
- P4.34
- P4.45
- P4.46
- P4.47
- P4.61
- P4.62
- P4.63
- T4.1
- T4.2
- T4.3