دانشگاه سمنان

Semnan University
Faculty of Mechanical Engineering


دانشکده مهندسی مکانیک

دانشکده مهندسی مکانیک

# درس استاتیک

# STATICS

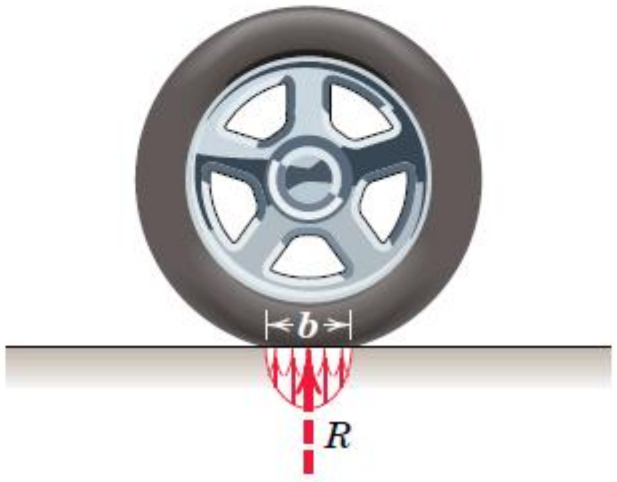
Chapter 5 - Distributed Forces

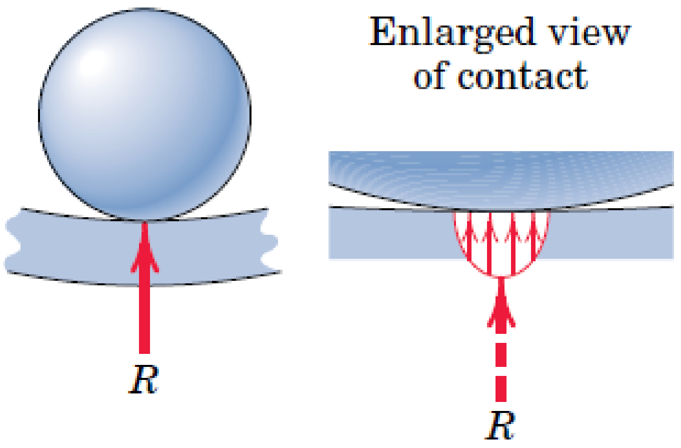
Class Lecture

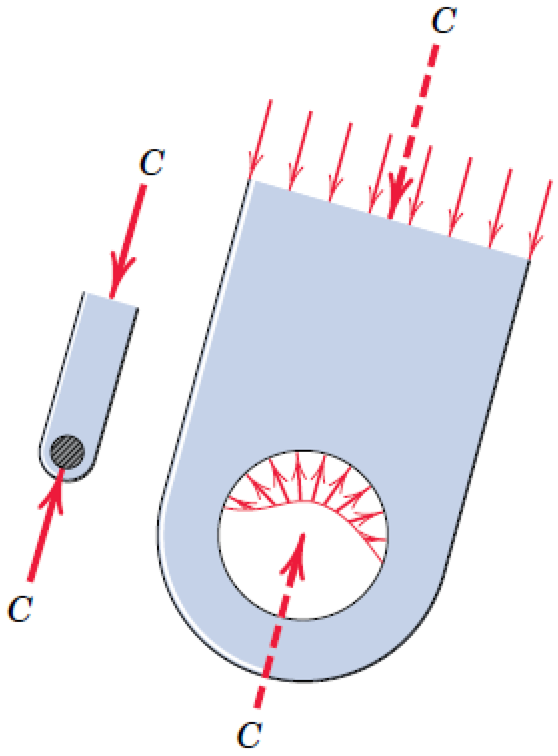
- ***CONTENTS:***

## 5.1 Introduction

- Actually, “concentrated” forces do not exist in the exact sense, since every external force applied mechanically to a body is distributed over a finite contact area, however small.



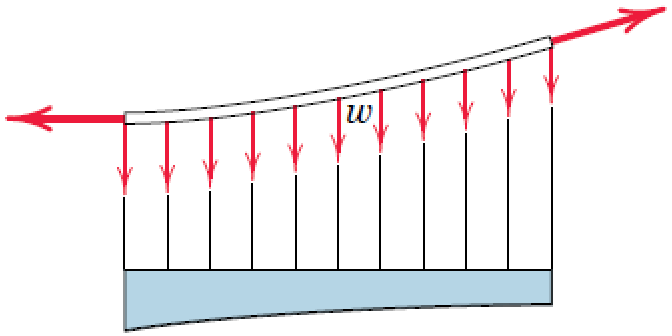



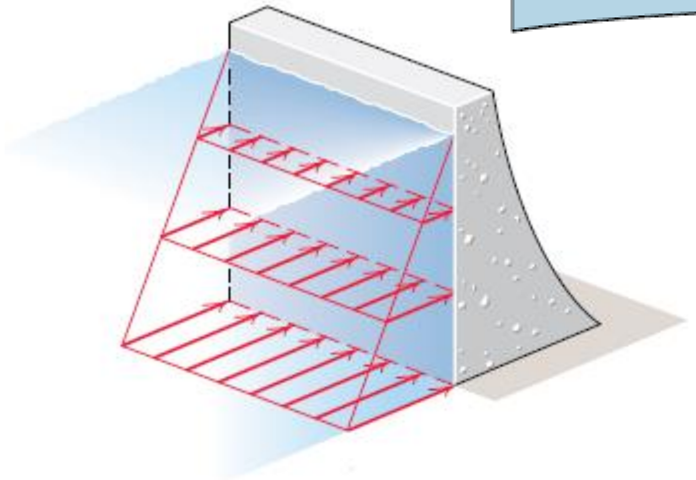
## 5.1 INTRODUCTION

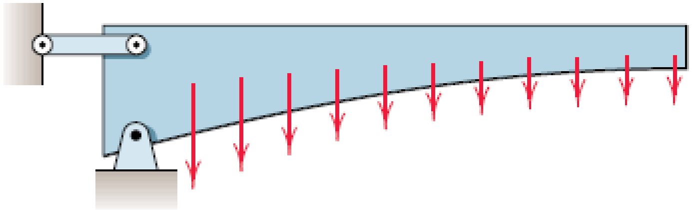
- When forces are applied over a region whose dimensions are not negligible compared with other pertinent dimensions, then we must account for the actual manner in which the force is distributed.
- We do this by summing the effects of the distributed force over the entire region using mathematical integration.
- This requires that we know the intensity of the force at any location.

# 5.1 Introduction

- There are three categories:
  - (1) Line Distribution
  - (2) Area Distribution
  - (3) Volume Distribution





## 5.1 INTRODUCTION

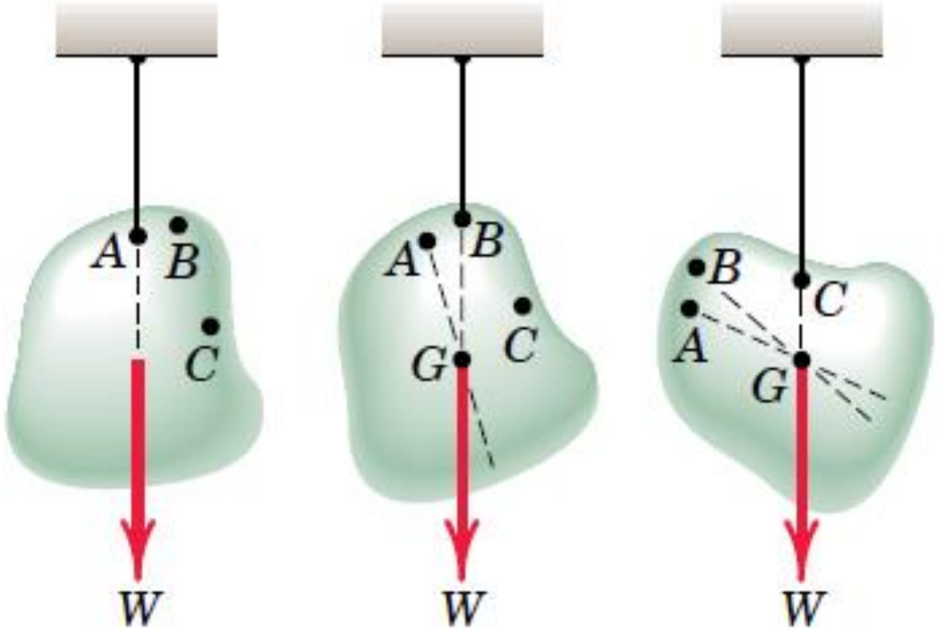
- Section A: CENTERS OF MASS AND CENTROIDS
  - Center of Mass
  - Centroids of Lines, Areas, and Volumes
  - Composite Bodies and Figures; Approximations
  - Theorems of Pappus

- Section B: SPECIAL TOPICS
  - Beams—External Effects
  - Beams—Internal Effects
  - Flexible Cables
  - Fluid Statics

## 5.2 CENTER OF MASS

- If we suspend the body from any point the body will be in equilibrium under the action of the cord tension and the resultant *W* of the gravitational forces acting on all particles of the body.

- If we repeat for other points, the center of gravity (CG) will be determined by intersection of these lines.

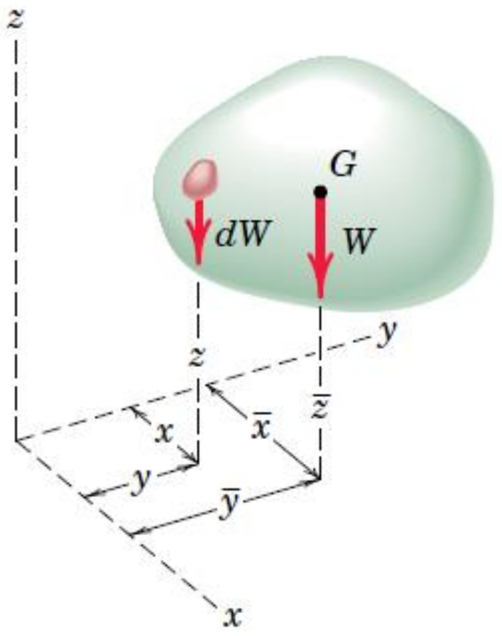


## 5.2 CENTER OF MASS

- Determining the Center of Gravity
  - The moment of the resultant gravitational force *W* about any axis equals the sum of the moments about the same axis of the gravitational forces *dW* acting on all particles.

$$\bar{x} = \frac{\int x\,dW}{W} \qquad \bar{y} = \frac{\int y\,dW}{W} \qquad \bar{z} = \frac{\int z\,dW}{W}$$

$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}$$



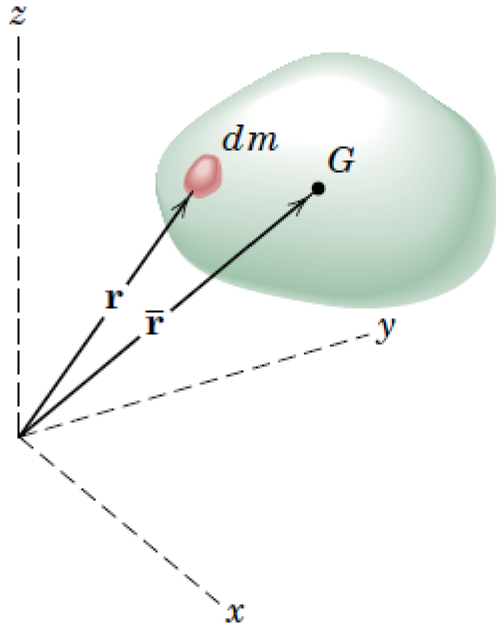
## 5.2 Center of Mass

- Determining the Center of Gravity
  - Vector form

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$$

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r}\, dm}{m}$$

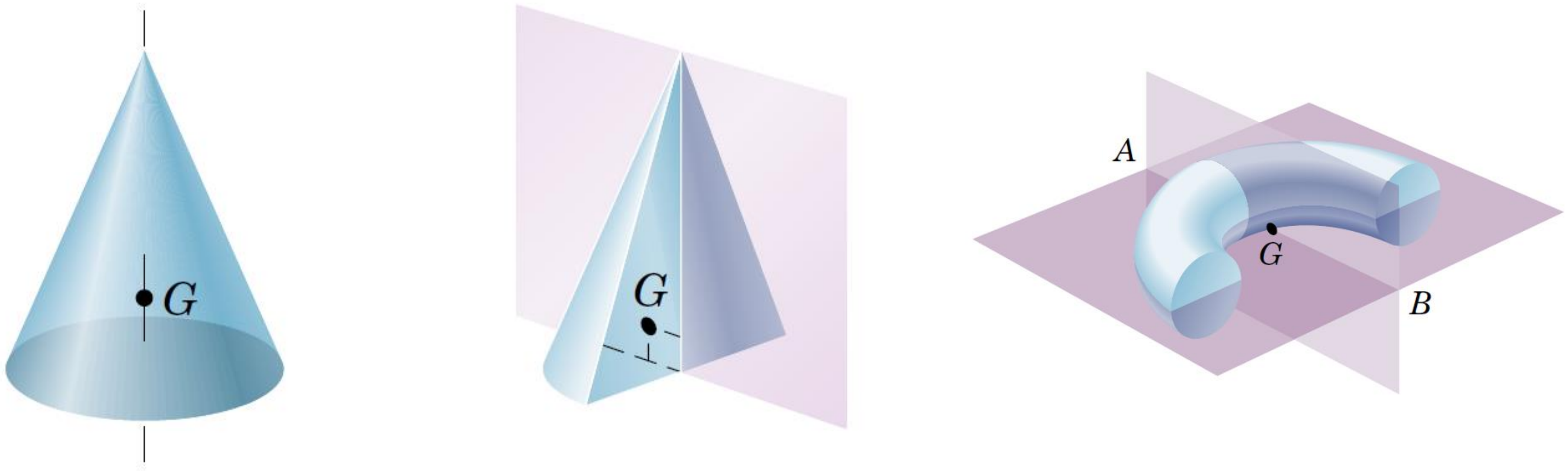



  - If ρ is not constant:

$$\bar{x} = \frac{\int x\rho\, dV}{\int \rho\, dV} \qquad \bar{y} = \frac{\int y\rho\, dV}{\int \rho\, dV} \qquad \bar{z} = \frac{\int z\rho\, dV}{\int \rho\, dV}$$

# 5.2 Center of Mass

- Using symmetry in CG determination

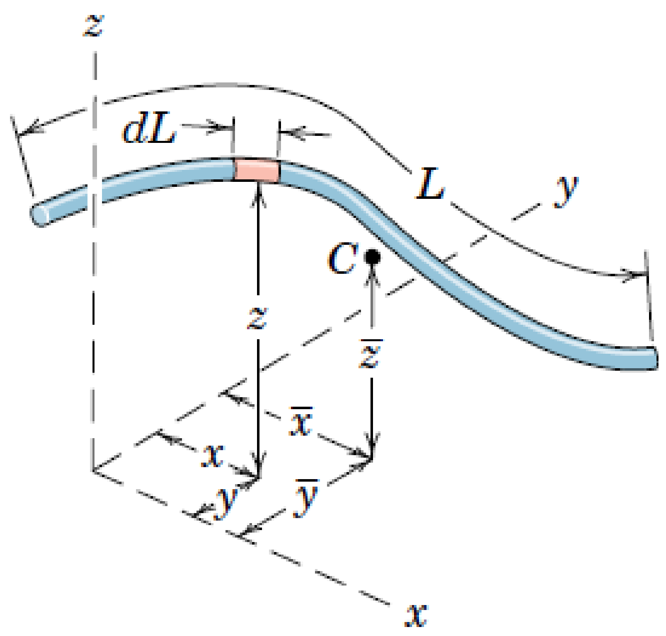


## 5.3 Centroids of Lines, Areas, and Volumes

- (1) Lines

$$dm = \rho A \, dL$$

$$\bar{x} = \frac{\int x \, dL}{L} \qquad \bar{y} = \frac{\int y \, dL}{L} \qquad \bar{z} = \frac{\int z \, dL}{L}$$

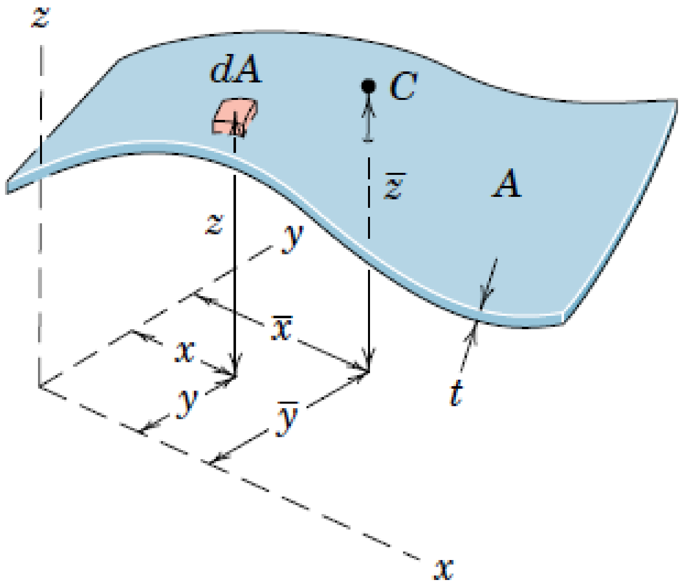


# 5.3 Centroids of Lines, Areas, and Volumes

- (2) Areas

$$dm = \rho t \, dA$$

$$\bar{x} = \frac{\int x \, dA}{A} \qquad \bar{y} = \frac{\int y \, dA}{A} \qquad \bar{z} = \frac{\int z \, dA}{A}$$

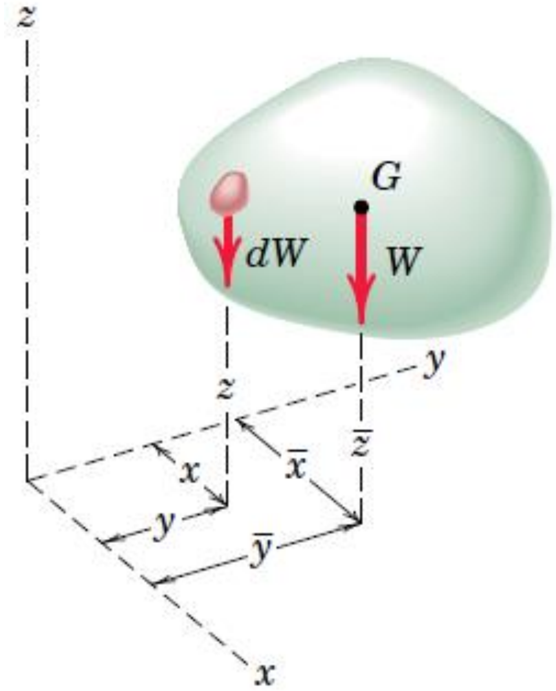


## 5.3 CENTROIDS OF LINES, AREAS, AND VOLUMES

- (3) Volumes

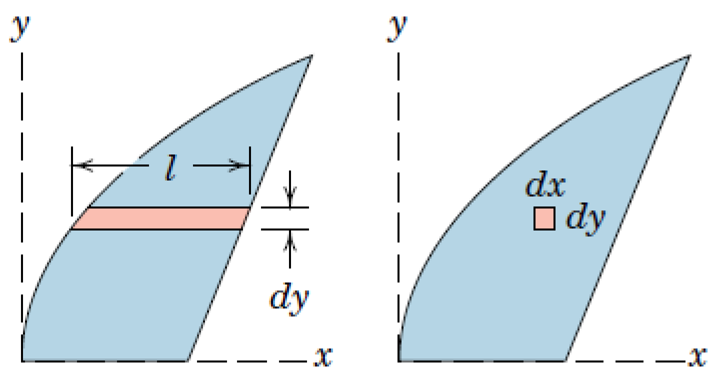
$$dm = \rho \, dV$$

$$\bar{x} = \frac{\int x \, dV}{V} \qquad \bar{y} = \frac{\int y \, dV}{V} \qquad \bar{z} = \frac{\int z \, dV}{V}$$

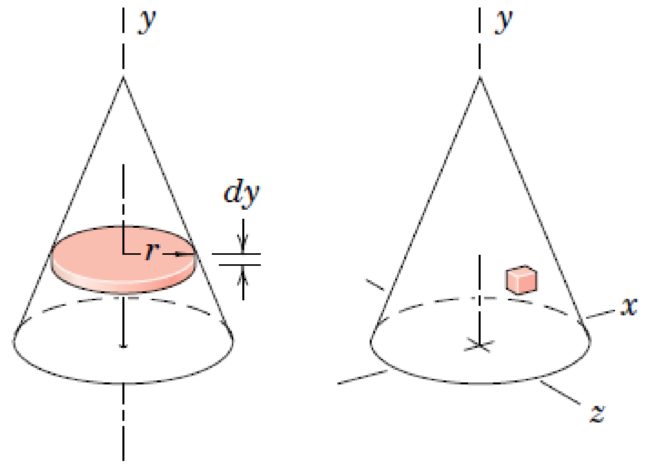


# 5.3 CENTROIDS OF LINES, AREAS, AND VOLUMES

- Integration guidelines:
  - (1) Order of Element.
    - ✓ Whenever possible, a first-order differential element should be selected.

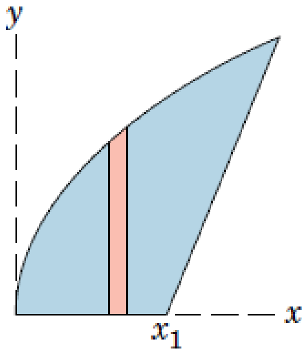


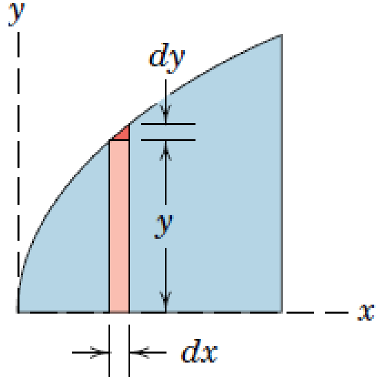

  - (2) Continuity.
    - ✓ Whenever possible, we choose an element which can be integrated in one continuous operation to cover the figure.



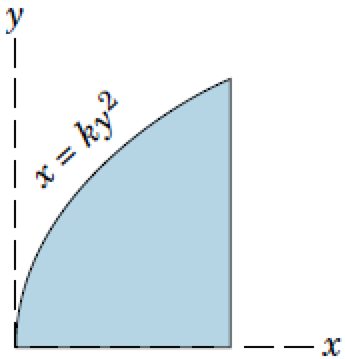
# 5.3 Centroids of Lines, Areas, and Volumes

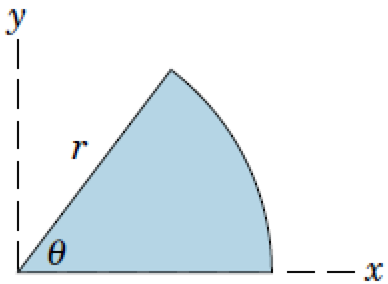
- Integration guidelines:
  - (3) Discarding Higher-Order Terms.
    - ✓ Higher-order terms may always be dropped compared with lower-order terms.



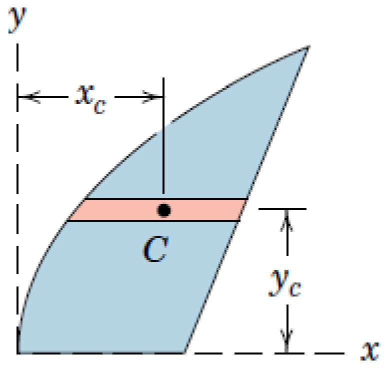

  - *(4) Choice of Coordinates.*
    - ✓ We choose the coordinate system which best matches the boundaries of the figure.

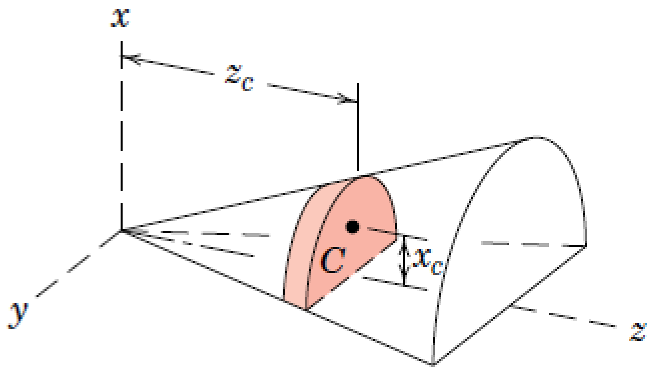


# 5.3 CENTROIDS OF LINES, AREAS, AND VOLUMES

- Integration guidelines:
  - (5) Centroidal Coordinate of Element
    - ✓ it is essential to use the *coordinate of the centroid of the element* for the moment arm in expressing the moment of the differential element.



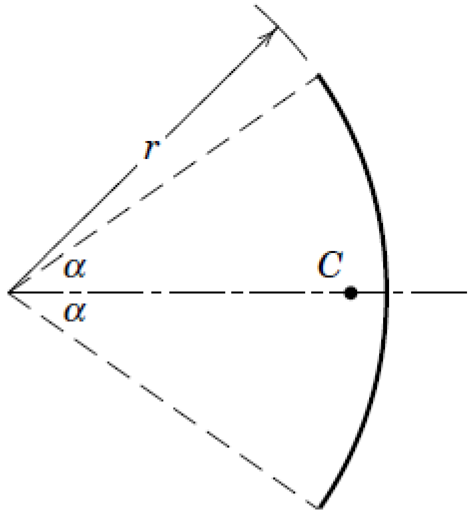




$$\bar{x} = \frac{\int x_c \, dA}{A} \qquad \bar{y} = \frac{\int y_c \, dA}{A} \qquad \bar{z} = \frac{\int z_c \, dA}{A}$$

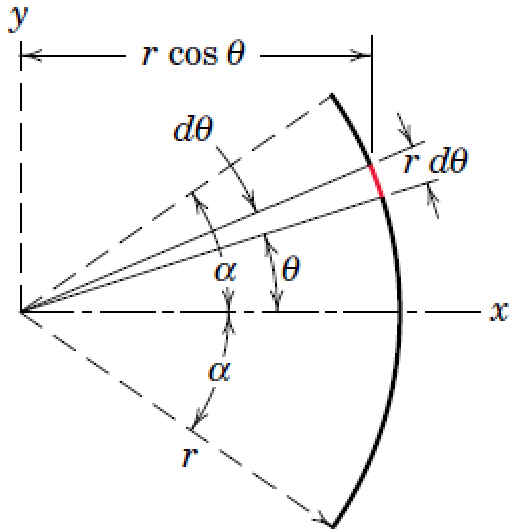
$$\bar{x} = \frac{\int x_c \, dV}{V} \qquad \bar{y} = \frac{\int y_c \, dV}{V} \qquad \bar{z} = \frac{\int z_c \, dV}{V}$$

## Sample Problem 5/1

***Centroid of a circular arc.*** Locate the centroid of a circular arc as shown in the figure.




Choosing the axis of symmetry as the $x$-axis makes $\bar{y} = 0$.



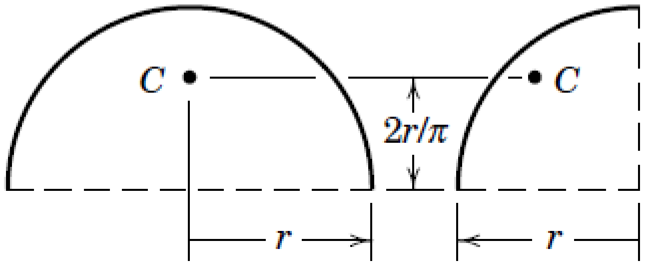

$dL = r\,d\theta$

$L = 2\alpha r$

$$[L\bar{x} = \int x\,dL] \quad \rightarrow \quad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r\cos\theta)\,r\,d\theta$$

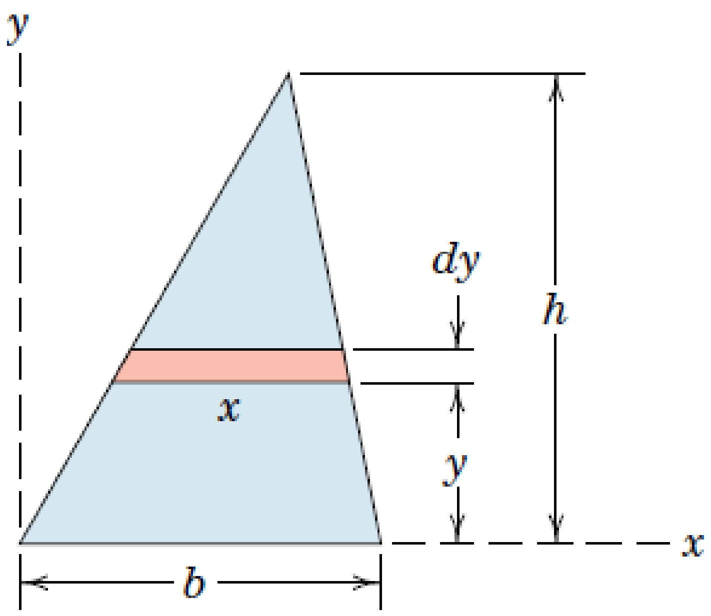
$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

$$\bar{x} = \frac{r\sin\alpha}{\alpha}$$



## Sample Problem 5/2

***Centroid of a triangular area.*** Determine the distance $\bar{h}$ from the base of a triangle of altitude $h$ to the centroid of its area.




$$dA = x\,dy$$

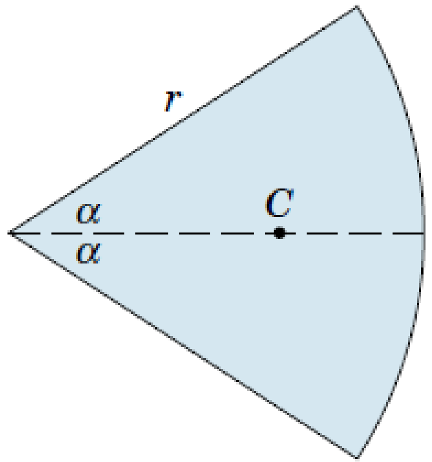
$$x/(h - y) = b/h$$

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y\,\frac{b(h-y)}{h}\,dy = \frac{bh^2}{6}$$

$$\bar{y} = \frac{h}{3}$$

# Sample Problem 5/3

***Centroid of the area of a circular sector.*** Locate the centroid of the area of a circular sector with respect to its vertex.




## *Solution I.*

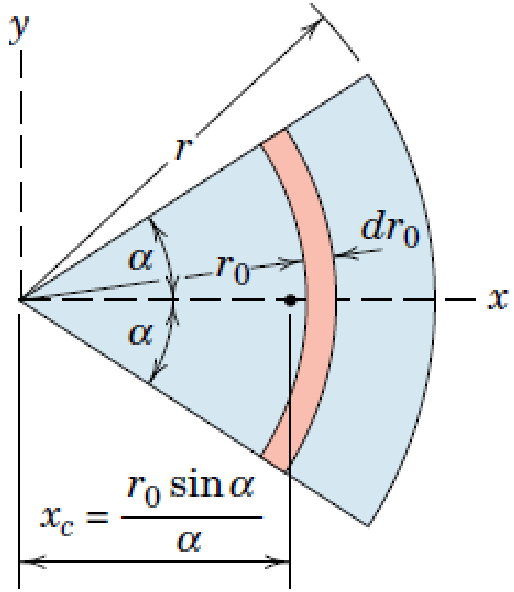
$$dA = 2r_0\alpha\, dr_0$$

$$[A\bar{x} = \int x_c\, dA]$$

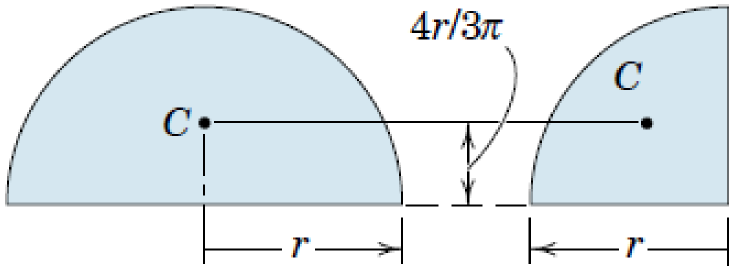
$$\frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha\, dr_0)$$

$$r^2\alpha\bar{x} = \frac{2}{3}r^3 \sin\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha}$$

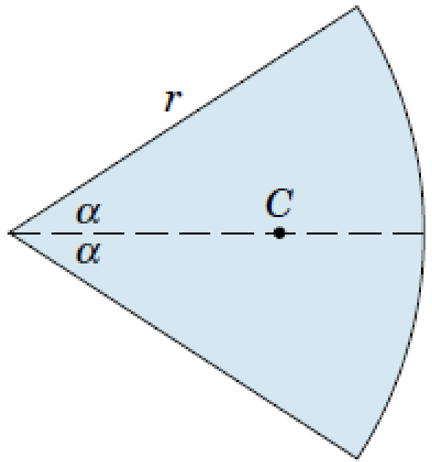



Solution I



# Sample Problem 5/3

***Centroid of the area of a circular sector.*** Locate the centroid of the area of a circular sector with respect to its vertex.




## *Solution II.*

$dA = (r/2)(r\,d\theta)$

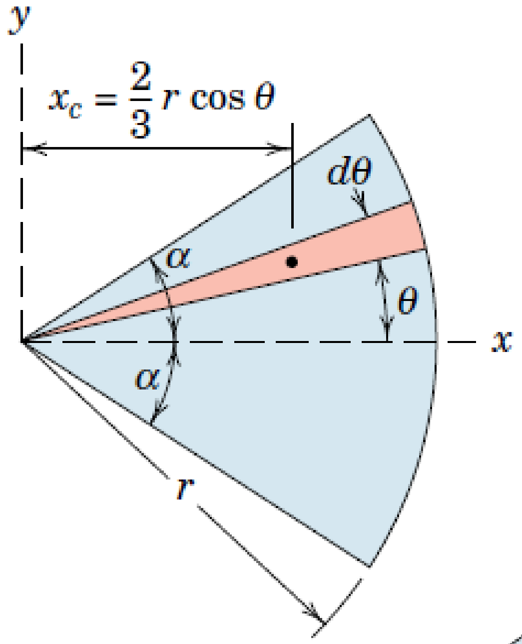
$x_c = \frac{2}{3}r\cos\theta$

$[A\bar{x} = \int x_c\,dA]$

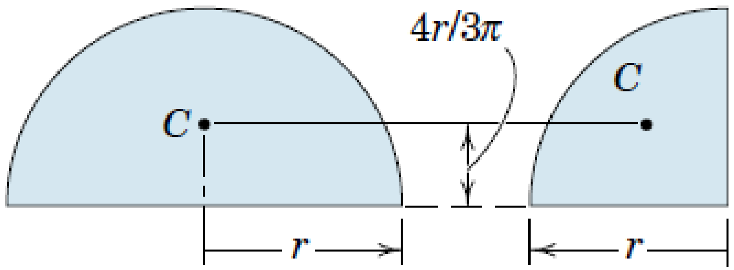
$$(r^2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} \left(\frac{2}{3}r\cos\theta\right)\left(\frac{1}{2}r^2\,d\theta\right)$$

$$r^2\alpha\bar{x} = \frac{2}{3}r^3\sin\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha}$$

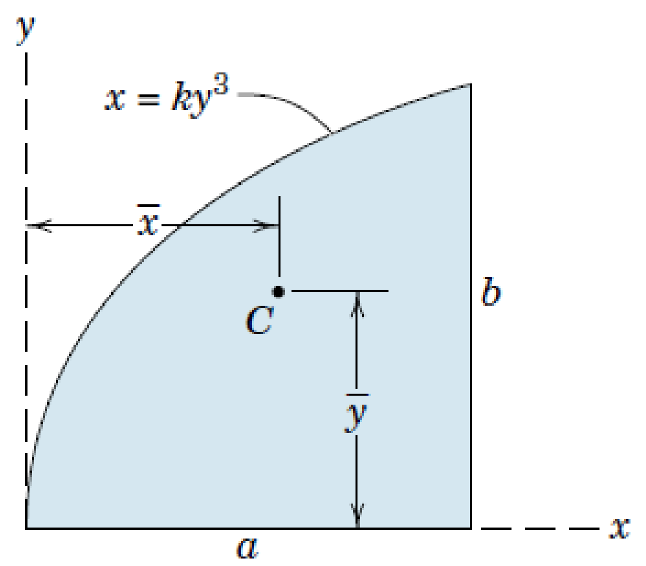



Solution II



## Sample Problem 5/4

Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.




### Solution I.

$dA = y\,dx$

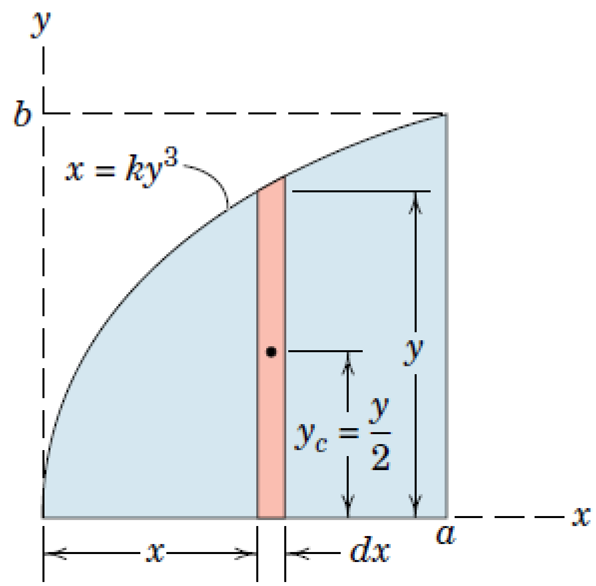
$[A\bar{x} = \int x_c\,dA] \rightarrow \bar{x}\int_0^a y\,dx = \int_0^a xy\,dx$

$y = (x/k)^{1/3}$ and $k = a/b^3$

$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7} \qquad \bar{x} = \frac{4}{7}a$

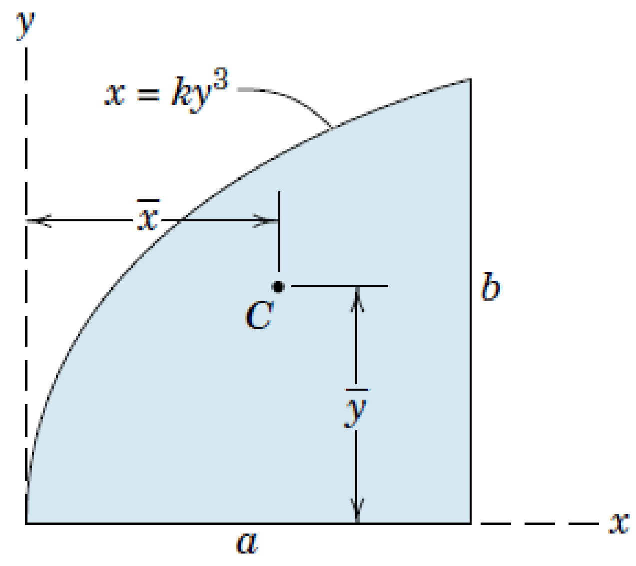
$[A\bar{y} = \int y_c\,dA] \rightarrow \frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right) y\,dx$

$\rightarrow \frac{3ab}{4}\bar{y} = \frac{3ab^2}{10} \qquad \bar{y} = \frac{2}{5}b$



## Sample Problem 5/4

Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.

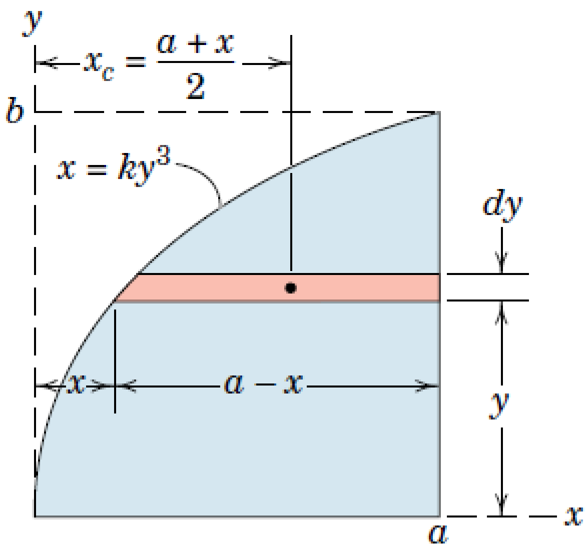



### Solution II.

$$x_c = x + \frac{1}{2}(a - x) = (a + x)/2$$

$$[A\bar{x} = \int x_c\, dA] \qquad \bar{x}\int_0^b (a - x)\, dy = \int_0^b \left(\frac{a + x}{2}\right)(a - x)\, dy$$

$$[A\bar{y} = \int y_c\, dA] \qquad \bar{y}\int_0^b (a - x)\, dy = \int_0^b y(a - x)\, dy$$

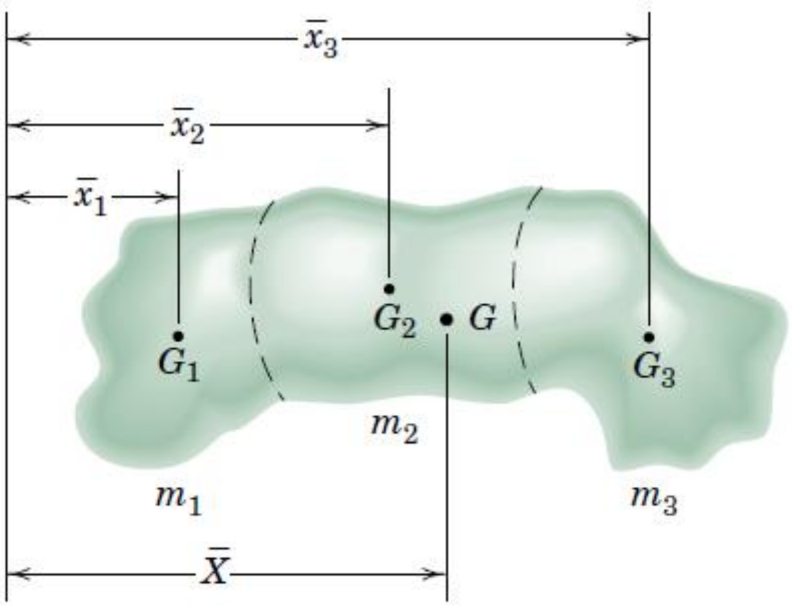


## 5.5 Composite Bodies and Figures; Approximations

- When a body or figure can be conveniently divided into several parts whose mass centers are easily determined, we use the principle of moments and treat each part as a finite element of the whole.

$$(m_1 + m_2 + m_3)\bar{X} = m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3$$

$$\bar{X} = \frac{\Sigma m\bar{x}}{\Sigma m} \qquad \bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m} \qquad \bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}$$



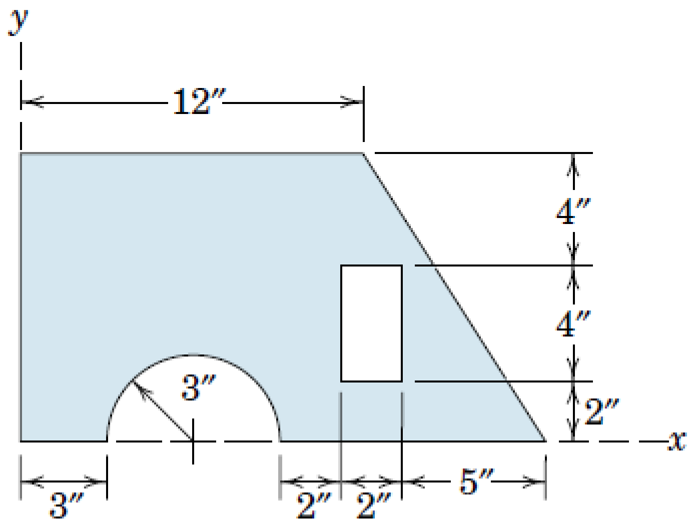
## Sample Problem 5/6

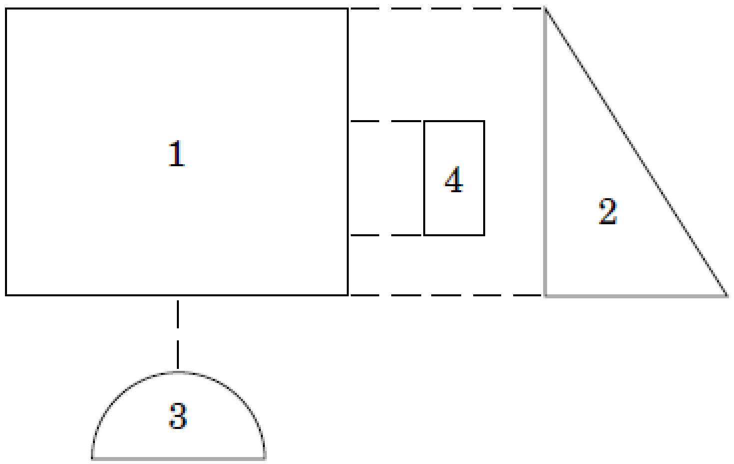
Locate the centroid of the shaded area.

| PART | A in.$^2$ | $\bar{x}$ in. | $\bar{y}$ in. | $\bar{x}A$ in.$^3$ | $\bar{y}A$ in.$^3$ |
|---|---|---|---|---|---|
| 1 | 120 | 6 | 5 | 720 | 600 |
| 2 | 30 | 14 | 10/3 | 420 | 100 |
| 3 | −14.14 | 6 | 1.273 | −84.8 | −18 |
| 4 | −8 | 12 | 4 | −96 | −32 |
| TOTALS | 127.9 | | | 959 | 650 |

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959}{127.9} = 7.50 \text{ in.}$$

$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650}{127.9} = 5.08 \text{ in.}$$



# 5.5 THEOREMS OF PAPPUS

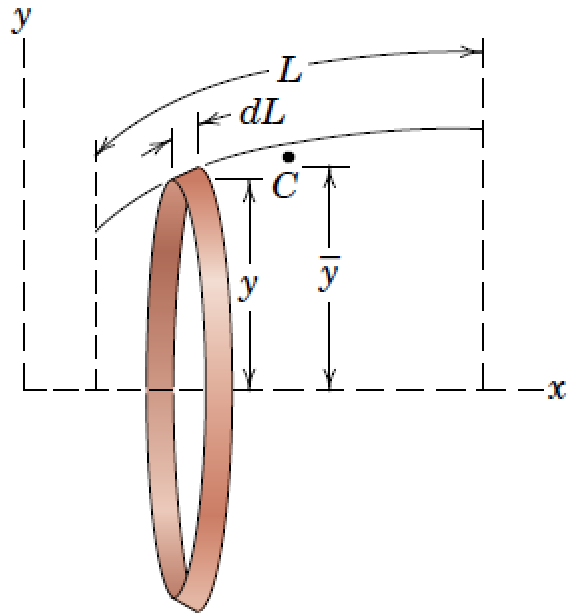
- Calculating the surface area generated by revolving a plane curve about a nonintersecting axis

$$dA = 2\pi y\, dL$$

$$A = 2\pi \int y\, dL$$

$$\bar{y}L = \int y\, dL$$

$$A = 2\pi \bar{y} L$$




- If a line is revolved through an angle θ less than 2π: (θ in radians)

$$A = \theta \bar{y} L$$

# 5.5 Theorems of Pappus

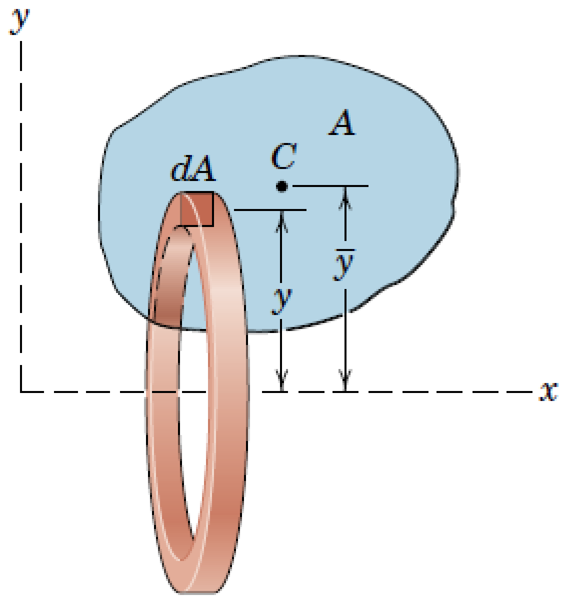
- Calculating the volume generated by revolving an area about a nonintersecting line in its plane

$$dV = 2\pi y\, dA$$

$$V = 2\pi \int y\, dA$$

$$\bar{y}A = \int y\, dA$$

$$\rightarrow \quad V = 2\pi\bar{y}A$$




- If an area is revolved through an angle θ less than 2π: (θ in radians)

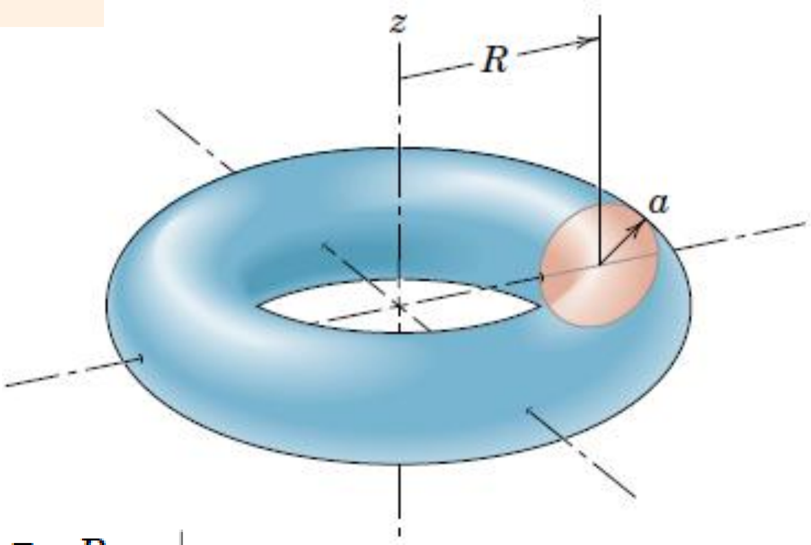
$$\rightarrow \quad V = \theta\bar{y}A$$

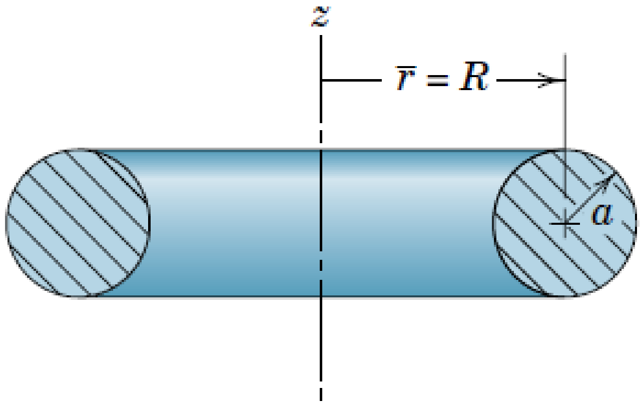
## Sample Problem 5/9

Determine the volume $V$ and surface area $A$ of the complete torus of circular cross section.

$$V = \theta \bar{r} A = 2\pi(R)(\pi a^2) = 2\pi^2 R a^2$$

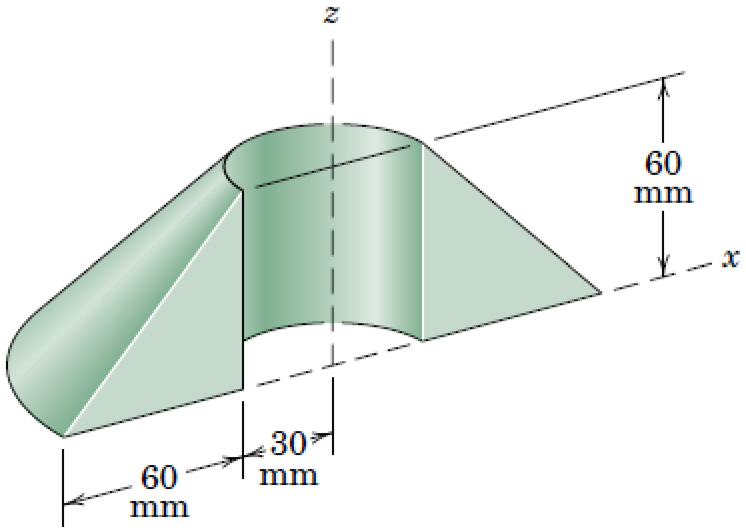
$$A = \theta \bar{r} L = 2\pi(R)(2\pi a) = 4\pi^2 R a$$



## Sample Problem 5/10

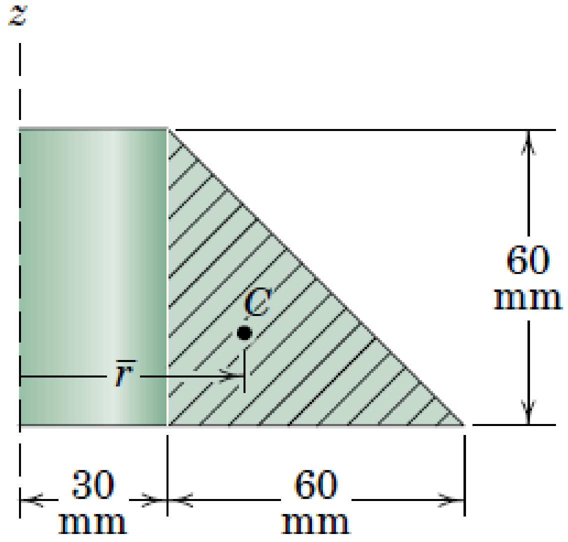
Calculate the volume $V$ of the solid generated by revolving the 60-mm right-triangular area through 180° about the $z$-axis. If this body were constructed of steel, what would be its mass $m$?




$$V = \theta \bar{r} A = \pi[30 + \tfrac{1}{3}(60)][\tfrac{1}{2}(60)(60)] = 2.83(10^5)\ \text{mm}^3$$

$$m = \rho V = \left[7830\ \frac{\text{kg}}{\text{m}^3}\right][2.83(10^5)\text{mm}^3]\left[\frac{1\ \text{m}}{1000\ \text{mm}}\right]^3$$

$$= 2.21\ \text{kg}$$



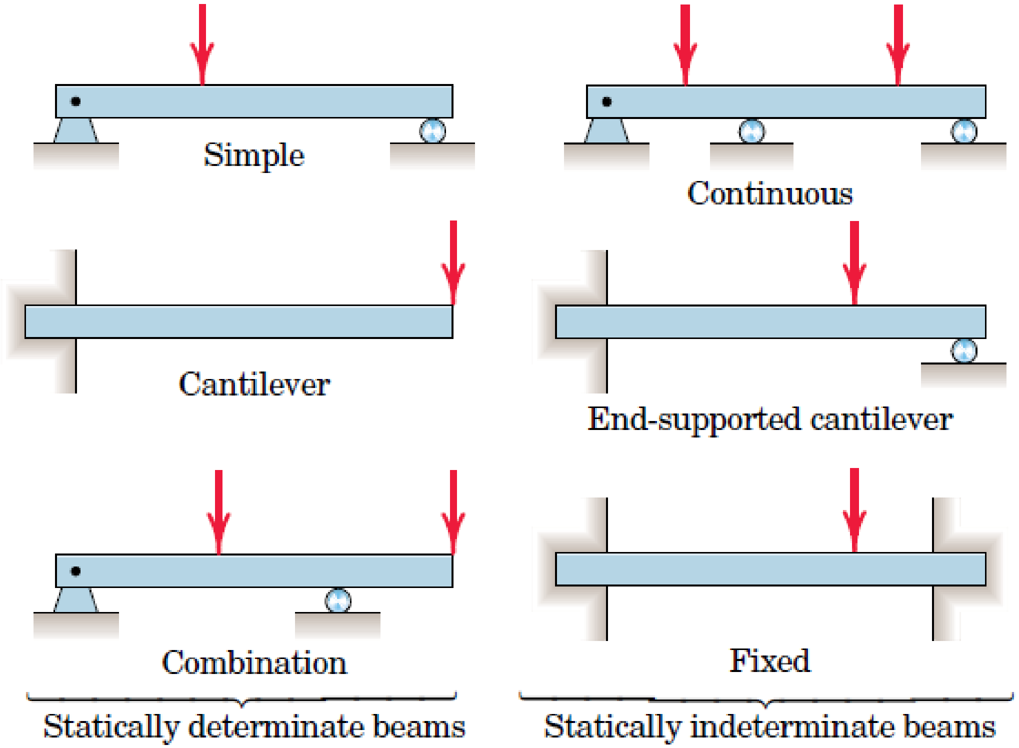
## 5.6 Beams - External Effects

- *Beams* are structural members which offer resistance to bending due to applied loads.
- Beams are undoubtedly the most important of all structural members, so it is important to understand the basic theory underlying their design.

- We must:
  - First, establish the equilibrium requirements of the beam as a whole and any portion of it considered separately.
  - Second, we must establish the relations between the resulting forces and the accompanying internal resistance of the beam to support these forces.

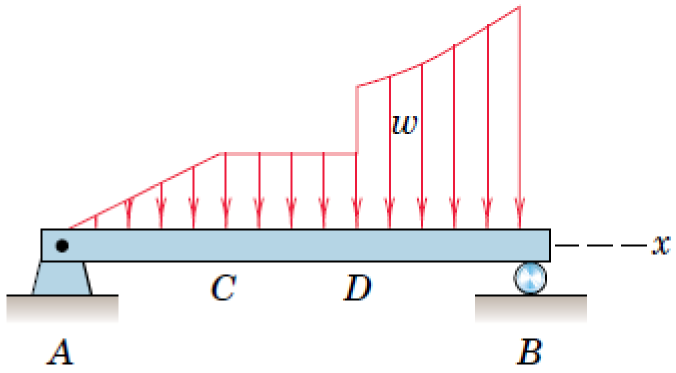
# 5.6 Beams - External Effects

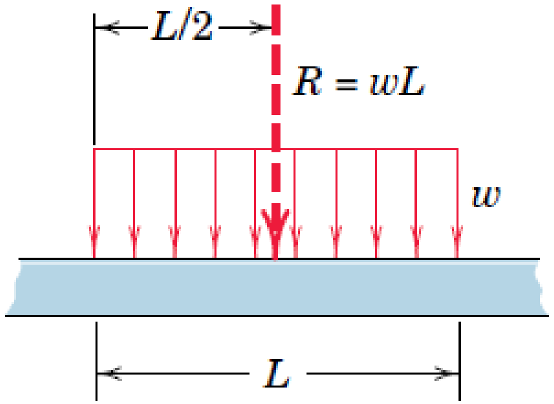
- Types of Beams:
  - *Statically determinate beams*
  - ✓ External support reactions can be calculated by the methods of statics alone are called.
  - *Statically indeterminate beams*
  - ✓ Has more supports than needed to provide equilibrium
  - ✓ Load-deformation properties should be considered to calculate external support reactions

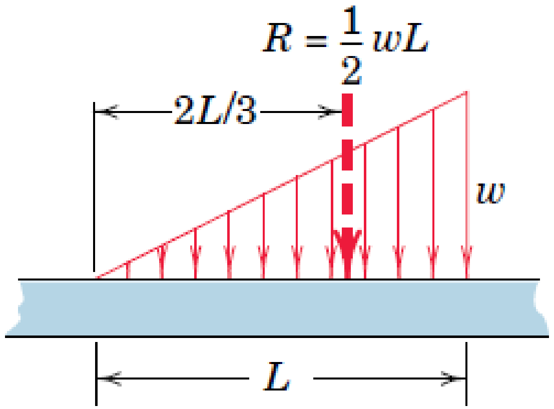


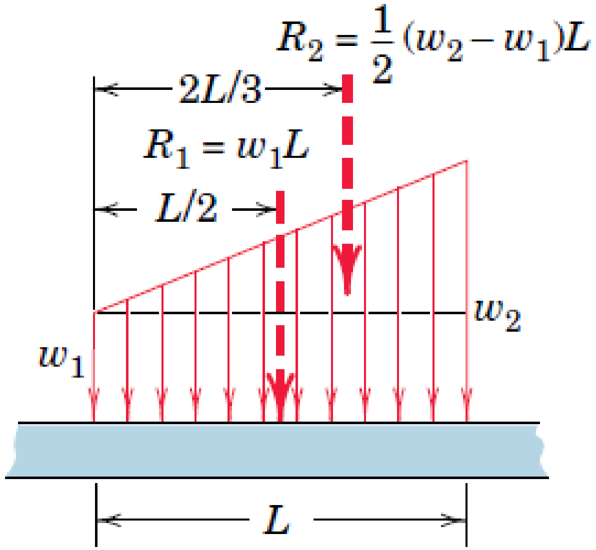
# 5.6 Beams - External Effects

- Distributed Loads
  - Broking to simple cases
    - ✓ Constant
    - ✓ Rectangular
    - ✓ Triangular



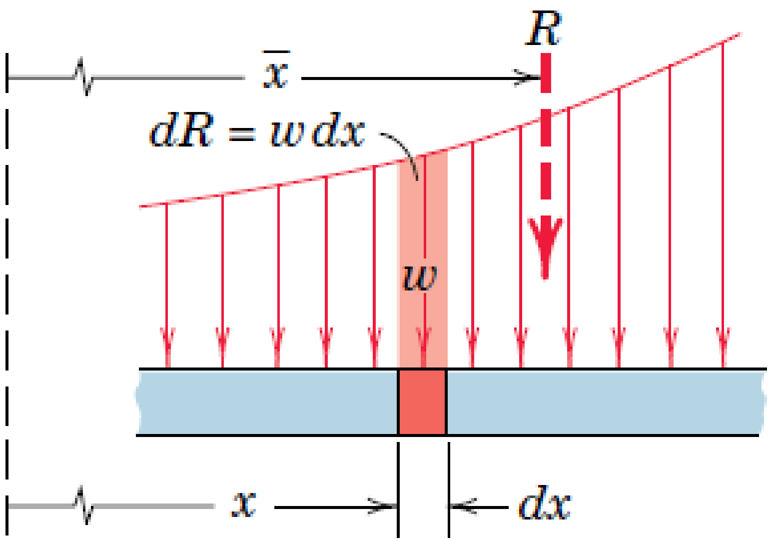



# 5.6 BEAMS - EXTERNAL EFFECTS

- Distributed Loads
  - General distribution

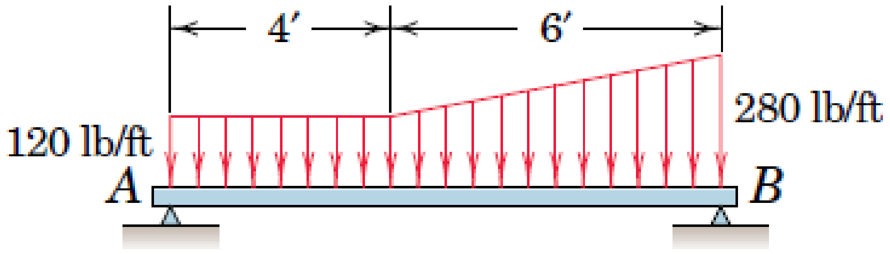
$$R = \int w \, dx$$

$$\bar{x} = \frac{\int xw \, dx}{R}$$



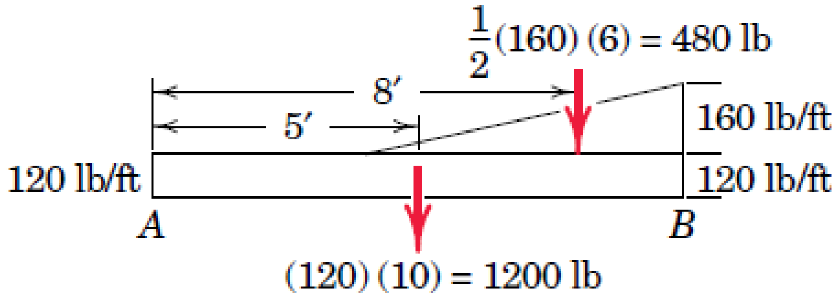
## Sample Problem 5/11

Determine the equivalent concentrated load(s) and external reactions for the simply supported beam which is subjected to the distributed load shown.



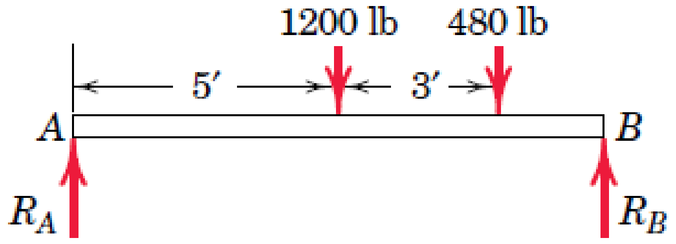

$[\Sigma M_A = 0]$ $\quad 1200(5) + 480(8) - R_B(10) = 0$

$$R_B = 984 \text{ lb}$$



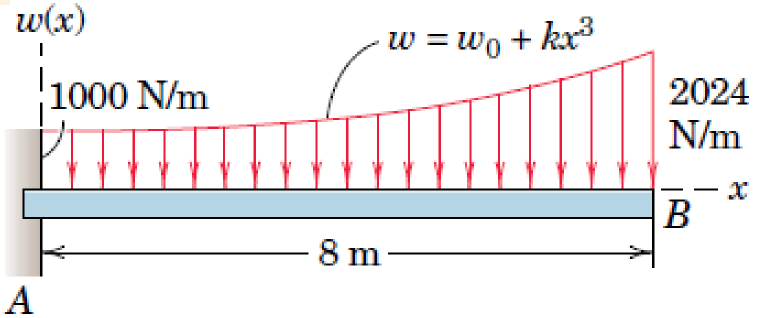

$[\Sigma M_B = 0]$ $\quad R_A(10) - 1200(5) - 480(2) = 0$

$$R_A = 696 \text{ lb}$$



# Sample Problem 5/12

Determine the reaction at the support $A$ of the loaded cantilever beam.

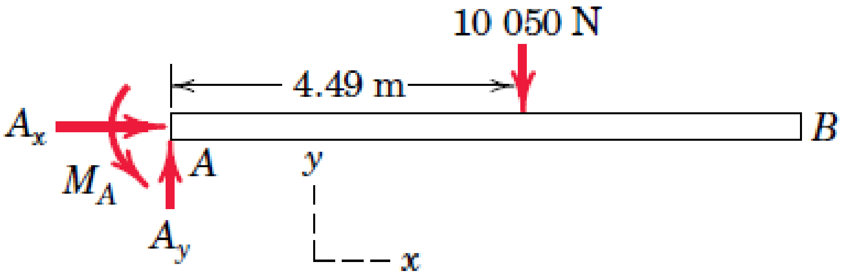



$w_0 = 1000$ N/m

$k = 2$ N/m$^4$

$$R = \int w\,dx = \int_0^8 (1000 + 2x^3)\,dx = \left(1000x + \frac{x^4}{2}\right)\Big|_0^8 = 10\,050 \text{ N}$$

$$\bar{x} = \frac{\int xw\,dx}{R} = \frac{1}{10\,050}\int_0^8 x(1000 + 2x^3)\,dx = \frac{1}{10\,050}\left(500x^2 + \tfrac{2}{5}x^5\right)\Big|_0^8 = 4.49 \text{ m}$$




$[\Sigma M_A = 0]$ $\quad M_A - (10\,050)(4.49) = 0$

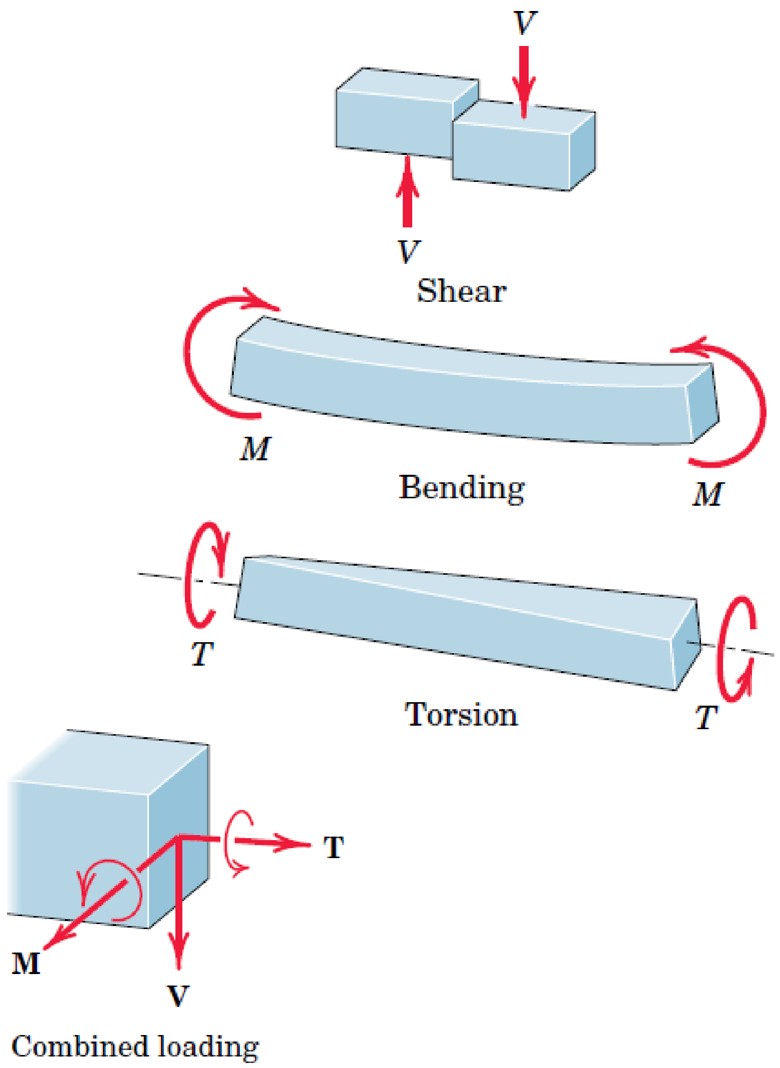
$M_A = 45\,100$ N·m

$[\Sigma F_y = 0]$ $\quad A_y = 10\,050$ N

# 5.7 BEAMS - INTERNAL EFFECTS

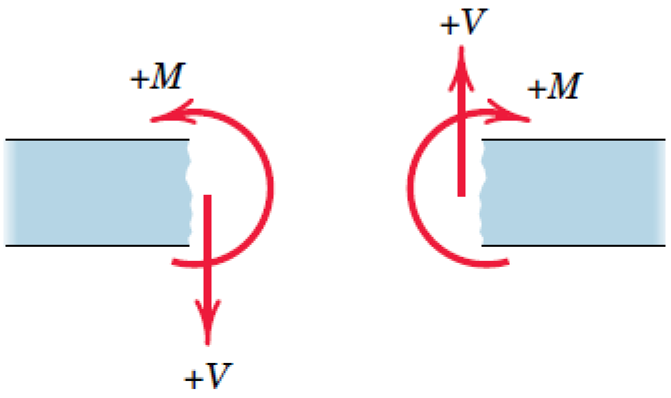
- In addition to supporting tension or compression, a beam can resist:
  - Shear
  - Bending
  - Torsion

✓ These effects represent the vector components of the resultant of the forces acting on a transverse section of the beam.

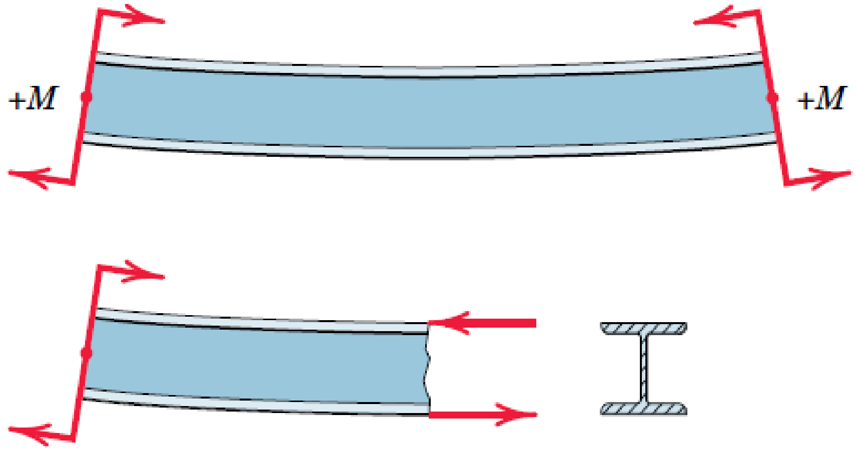


# 5.7 Beams - Internal Effects

- The conventions for positive values of shear *V* and bending moment *M* :




- Physical interpretation of the bending couple *M* :



# 5.7 Beams - Internal Effects

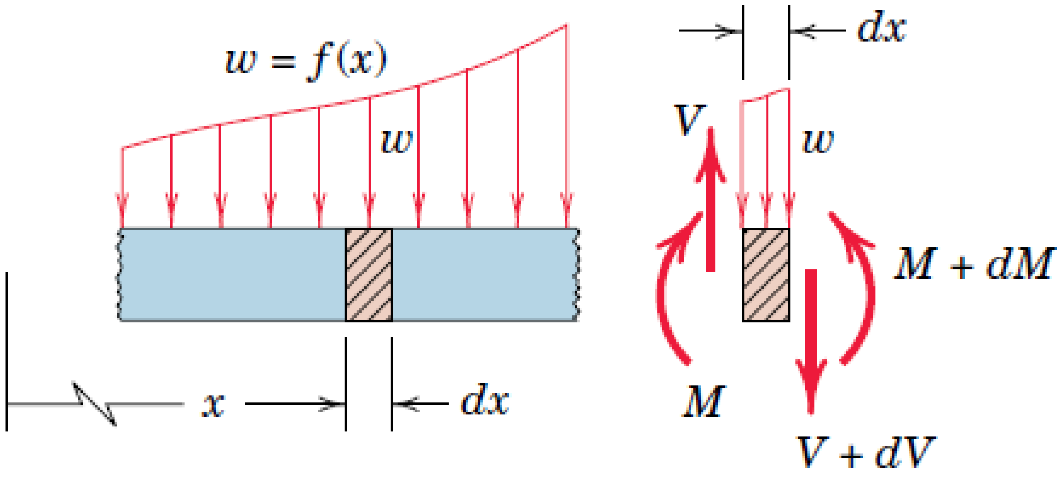
- General Loading, Shear, and Moment Relationships
  - Sum of the vertical forces = 0

$$V - w\,dx - (V + dV) = 0$$

$$w = -\frac{dV}{dx}$$

$$\int_{V_0}^{V} dV = -\int_{x_0}^{x} w\,dx$$

$V = V_0$ + (the negative of the area under the loading curve from $x_0$ to $x$)



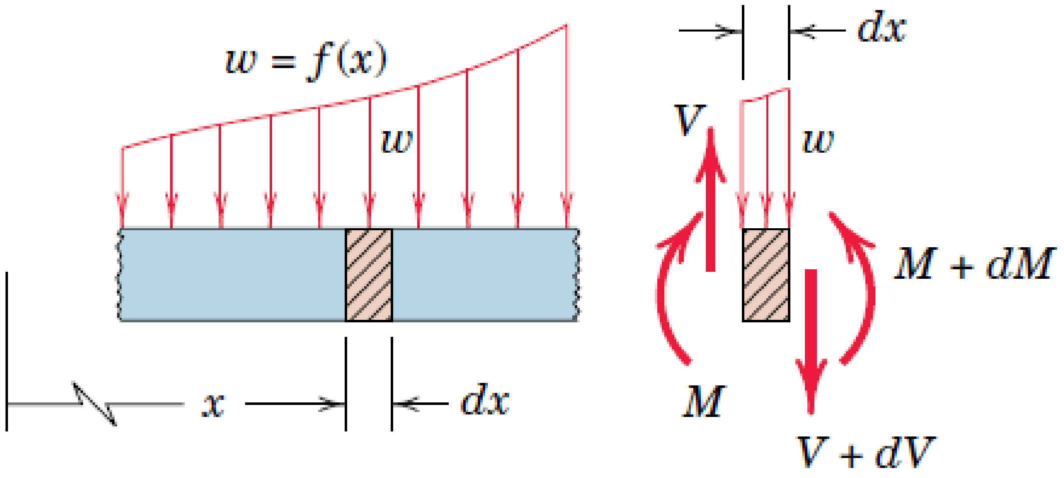
# 5.7 BEAMS - INTERNAL EFFECTS

- General Loading, Shear, and Moment Relationships
  - Sum of the moments about left side = 0

$$M + w\,dx\,\frac{dx}{2} + (V + dV)\,dx - (M + dM) = 0$$

terms $w(dx)^2/2$ and $dV\,dx$ may be dropped

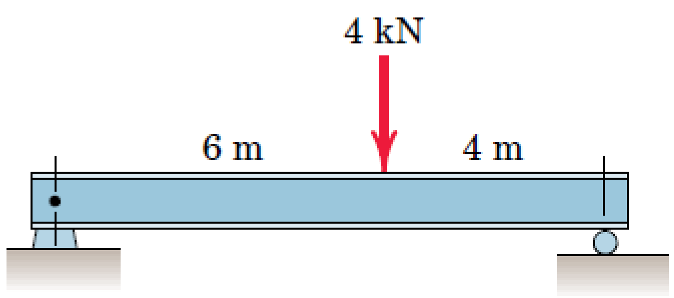
$$\rightarrow \quad V = \frac{dM}{dx} \qquad \frac{d^2M}{dx^2} = -w$$



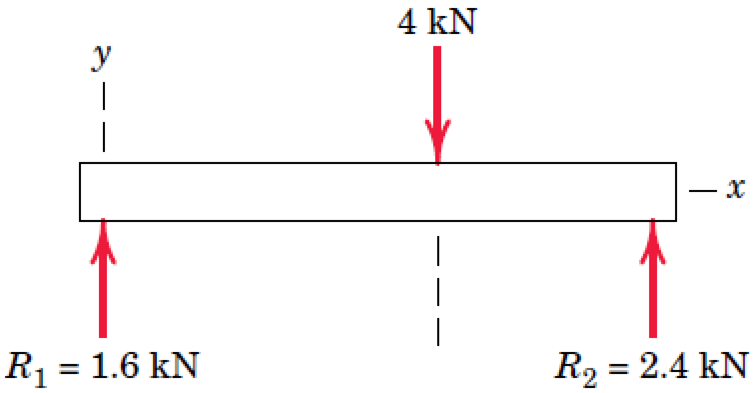

$$\int_{M_0}^{M} dM = \int_{x_0}^{x} V\,dx \quad \rightarrow \quad M = M_0 + (\text{area under the shear diagram from } x_0 \text{ to } x)$$

# Sample Problem 5/13

Determine the shear and moment distributions produced in the simple beam by the 4-kN concentrated load.




$R_1 = 1.6 \text{ kN} \qquad R_2 = 2.4 \text{ kN}$

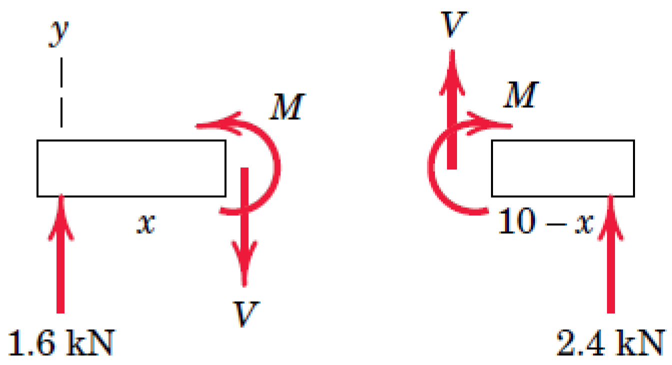



$[\Sigma F_y = 0] \qquad 1.6 - V = 0 \qquad V = 1.6 \text{ kN}$

$[\Sigma M_{R_1} = 0] \qquad M - 1.6x = 0 \qquad M = 1.6x$

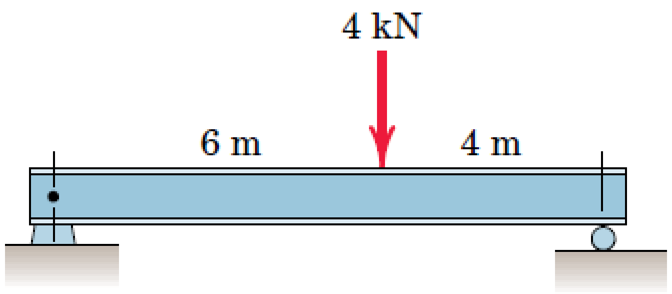
$[\Sigma F_y = 0] \qquad V + 2.4 = 0 \qquad V = -2.4 \text{ kN}$

$[\Sigma M_{R_2} = 0] \qquad -(2.4)(10 - x) + M = 0 \qquad M = 2.4(10 - x)$



## Sample Problem 5/13

Determine the shear and moment distributions produced in the simple beam by the 4-kN concentrated load.



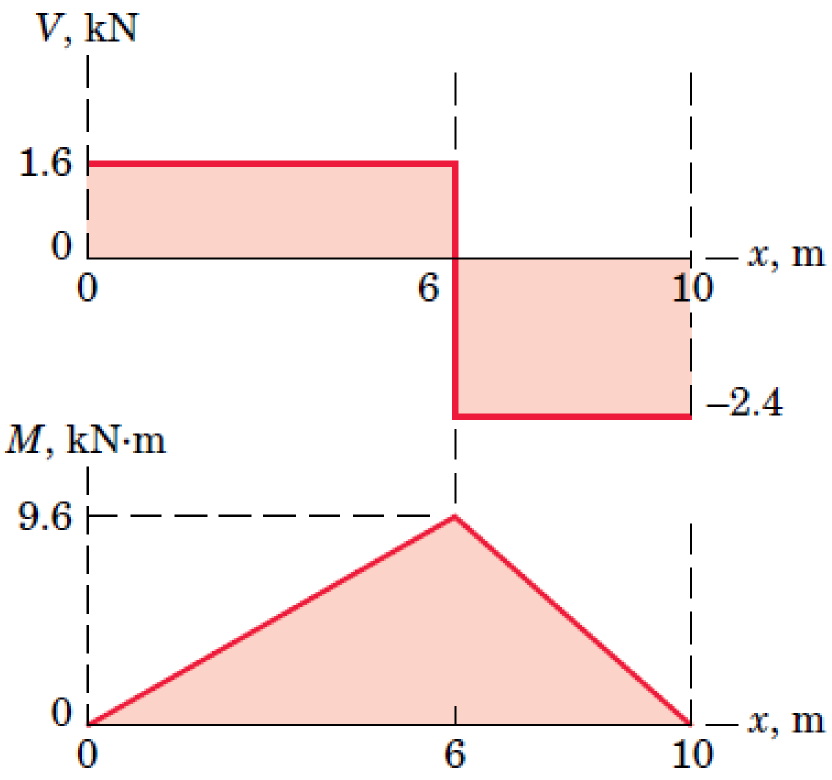

$R_1 = 1.6$ kN $\qquad R_2 = 2.4$ kN

$[\Sigma F_y = 0] \qquad 1.6 - V = 0 \qquad V = 1.6$ kN

$[\Sigma M_{R_1} = 0] \qquad M - 1.6x = 0 \qquad M = 1.6x$

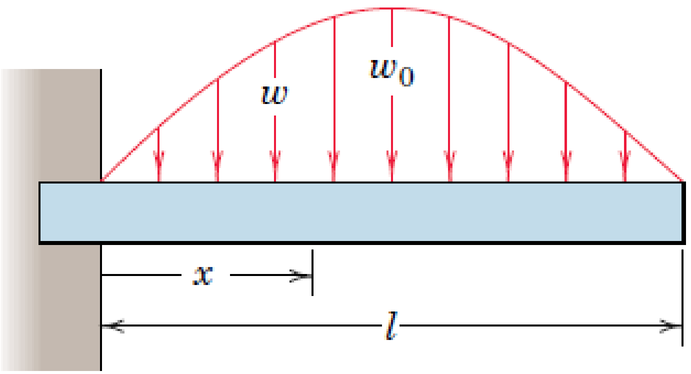
$[\Sigma F_y = 0] \qquad V + 2.4 = 0 \qquad V = -2.4$ kN

$[\Sigma M_{R_2} = 0] \qquad -(2.4)(10 - x) + M = 0 \qquad M = 2.4(10 - x)$



## Sample Problem 5/14

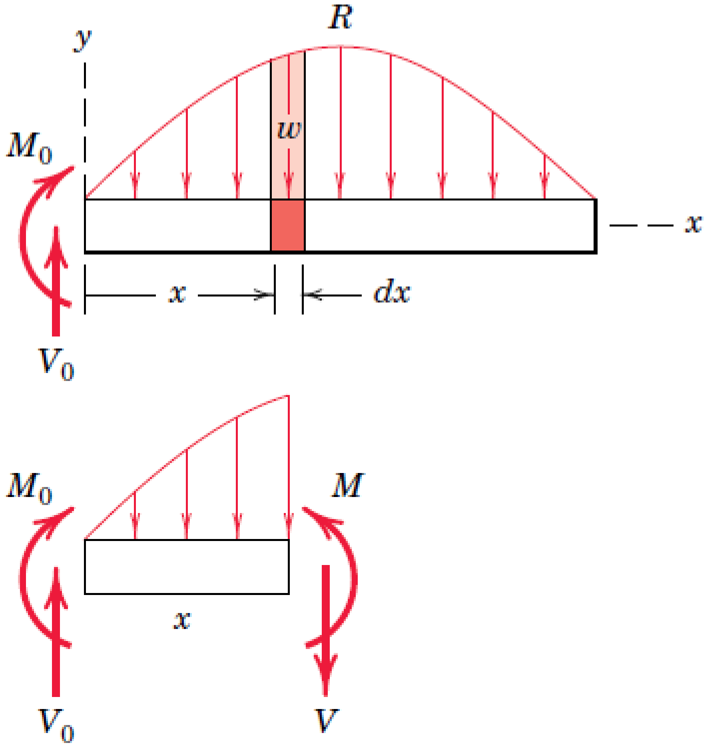
The cantilever beam is subjected to the load intensity (force per unit length) which varies as $w = w_0 \sin(\pi x/l)$. Determine the shear force $V$ and bending moment $M$ as functions of the ratio $x/l$.




$$[\Sigma F_y = 0] \qquad V_0 - \int_0^l w\,dx = 0 \qquad V_0 = \int_0^l w_0 \sin\frac{\pi x}{l}\,dx = \frac{2w_0 l}{\pi}$$

$$[\Sigma M = 0] \qquad -M_0 - \int_0^l x(w\,dx) = 0 \qquad M_0 = -\int_0^l w_0 x \sin\frac{\pi x}{l}\,dx$$

$$M_0 = \frac{-w_0 l^2}{\pi^2}\left[\sin\frac{\pi x}{l} - \frac{\pi x}{l}\cos\frac{\pi x}{l}\right]_0^l = -\frac{w_0 l^2}{\pi}$$



$$[dV = -w\,dx] \qquad \int_{V_0}^{V} dV = -\int_0^x w_0 \sin\frac{\pi x}{l}\,dx$$

$$V - V_0 = \left[\frac{w_0 l}{\pi}\cos\frac{\pi x}{l}\right]_0^x \qquad V - \frac{2w_0 l}{\pi} = \frac{w_0 l}{\pi}\left(\cos\frac{\pi x}{l} - 1\right)$$

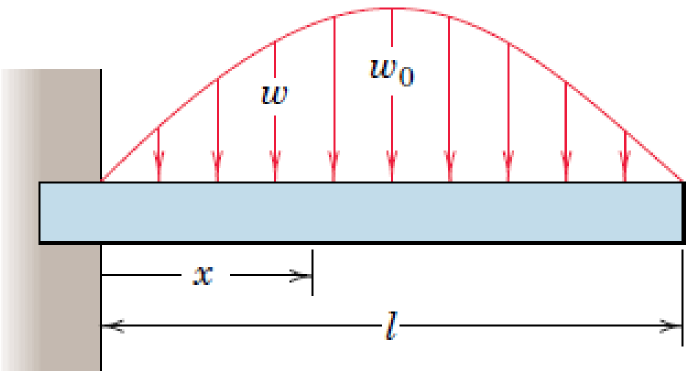
$$\frac{V}{w_0 l} = \frac{1}{\pi}\left(1 + \cos\frac{\pi x}{l}\right)$$

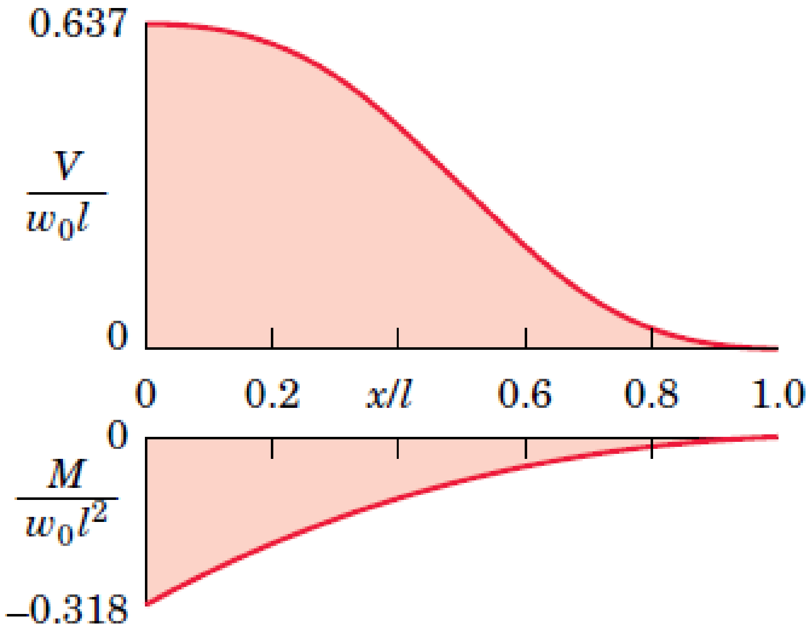
$$[dM = V\,dx] \qquad \int_{M_0}^{M} dM = \int_0^x \frac{w_0 l}{\pi}\left(1 + \cos\frac{\pi x}{l}\right)dx$$

$$M - M_0 = \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l}\right]_0^x$$

$$M = -\frac{w_0 l^2}{\pi} + \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l} - 0\right]$$

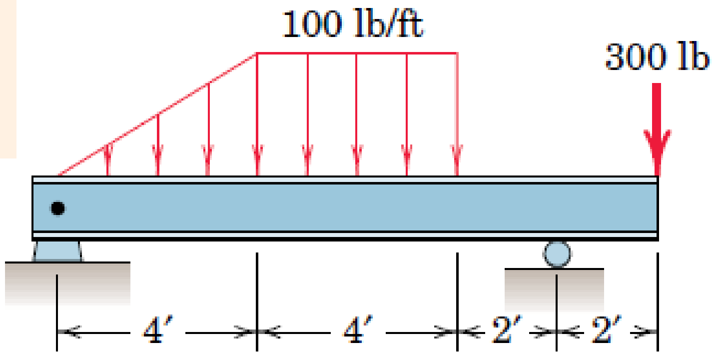
$$\frac{M}{w_0 l^2} = \frac{1}{\pi}\left(\frac{x}{l} - 1 + \frac{1}{\pi}\sin\frac{\pi x}{l}\right)$$



## Sample Problem 5/15

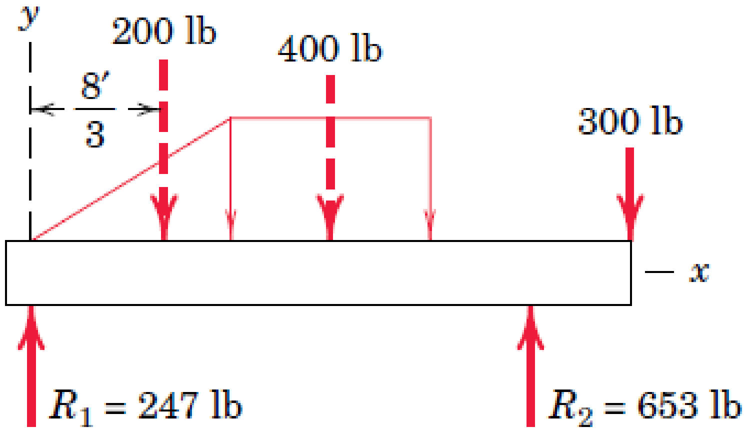
Draw the shear-force and bending-moment diagrams for the loaded beam and determine the maximum moment $M$ and its location $x$ from the left end.

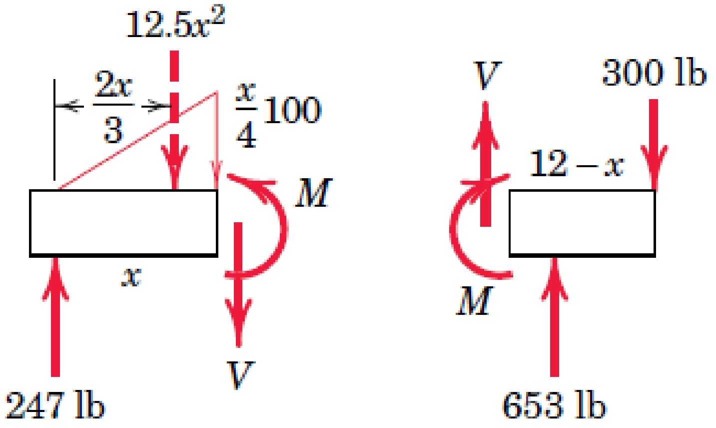



$0 < x < 4$ ft.

$[\Sigma F_y = 0]$ $\qquad V = 247 - 12.5x^2$

$[\Sigma M = 0]$ $\qquad M + (12.5x^2)\dfrac{x}{3} - 247x = 0 \qquad M = 247x - 4.17x^3$



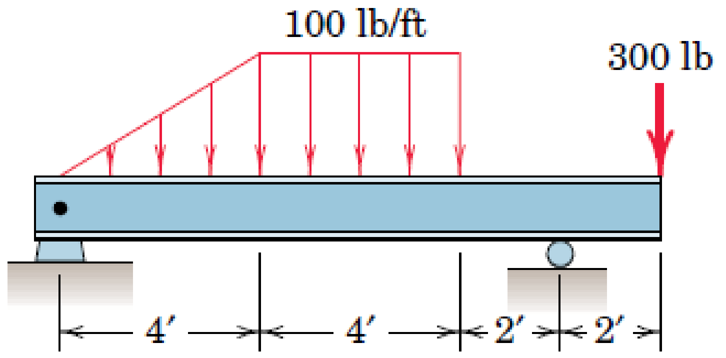



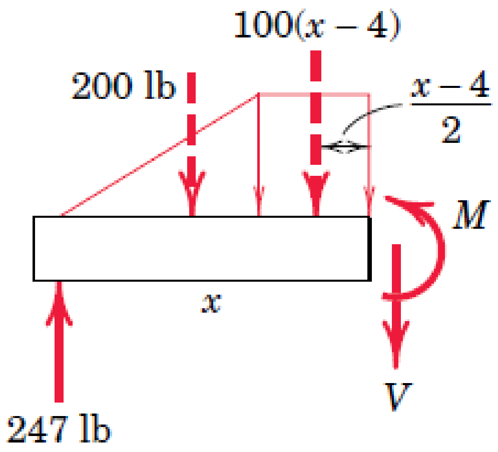
$4 < x < 8$ ft

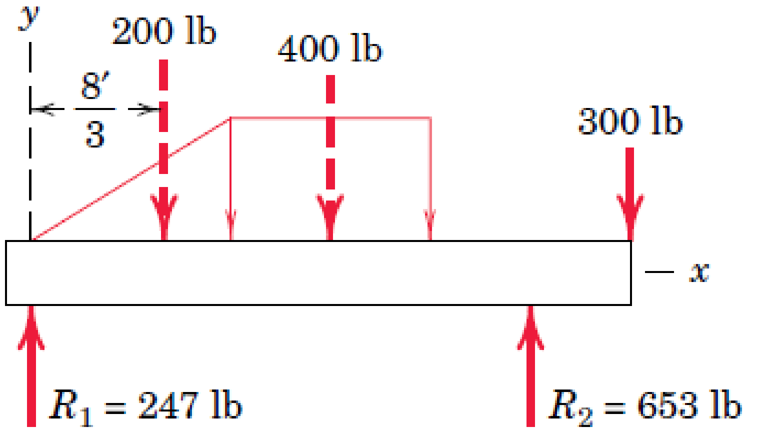
$$[\Sigma F_y = 0] \quad V + 100(x - 4) + 200 - 247 = 0 \qquad V = 447 - 100x$$

$$[\Sigma M = 0] \quad M + 100(x - 4)\frac{x-4}{2} + 200[x - \tfrac{2}{3}(4)] - 247x = 0$$

$$M = -267 + 447x - 50x^2$$



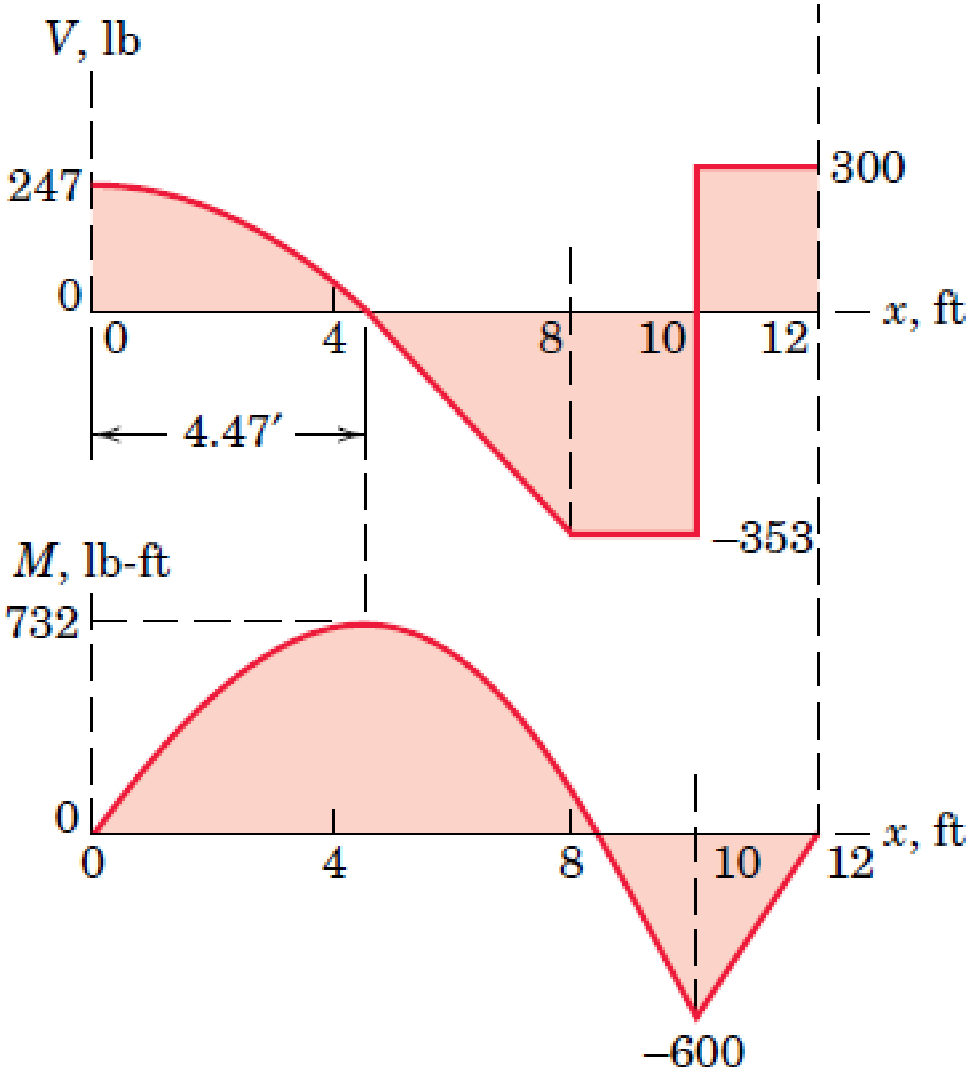



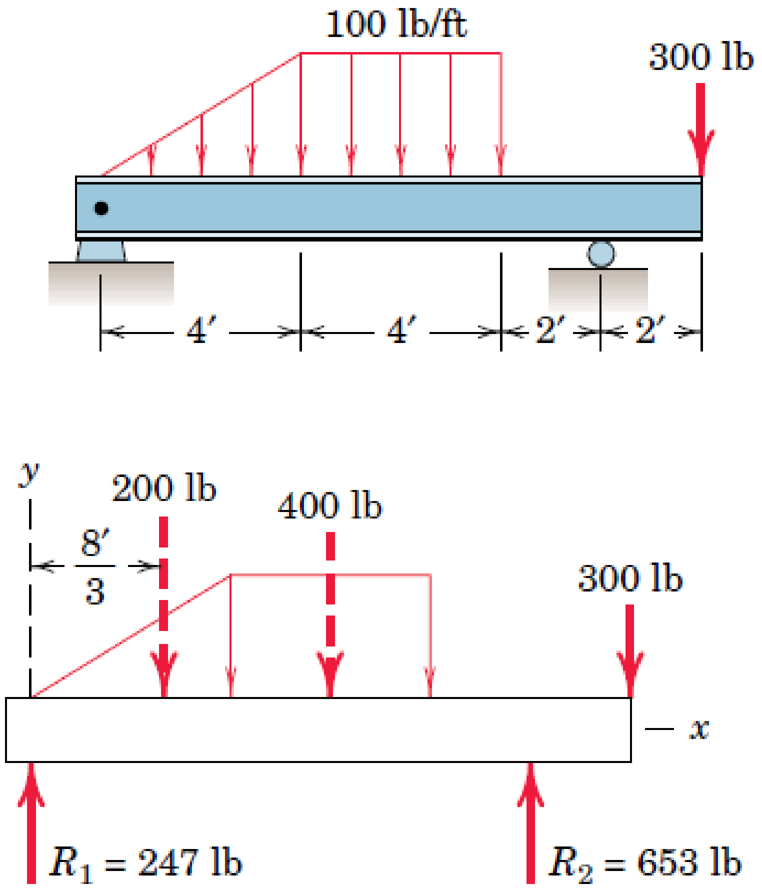




The analysis of the remainder of the beam

$8 < x < 10$ ft

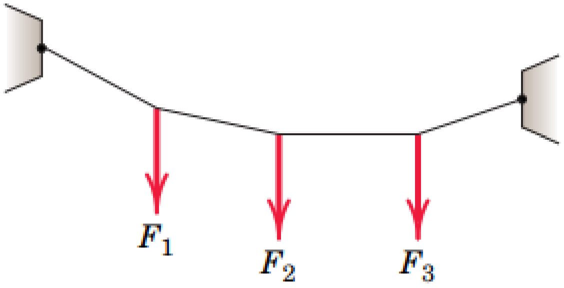
$$V = -353 \text{ lb} \qquad \text{and} \qquad M = 2930 - 353x$$

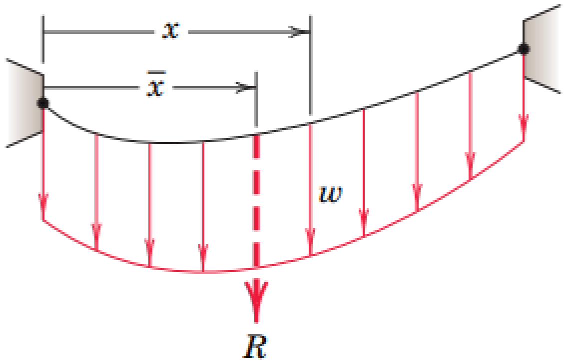

V, lb
247
0
300
0
4
8
10
12
x, ft
4.47′
−353
M, lb-ft
732
0
0
4
8
10
12
x, ft
−600


100 lb/ft
300 lb
4′
4′
2′
2′
y
200 lb
400 lb
8′
3
300 lb
x
R1 = 247 lb
R2 = 653 lb

# 5.8 FLEXIBLE CABLES

- One important type of structural member is the flexible cable which is used in suspension bridges, transmission lines, …
- Flexible cables may support a series of distinct concentrated loads, or they may support loads continuously distributed over the length of the cable.



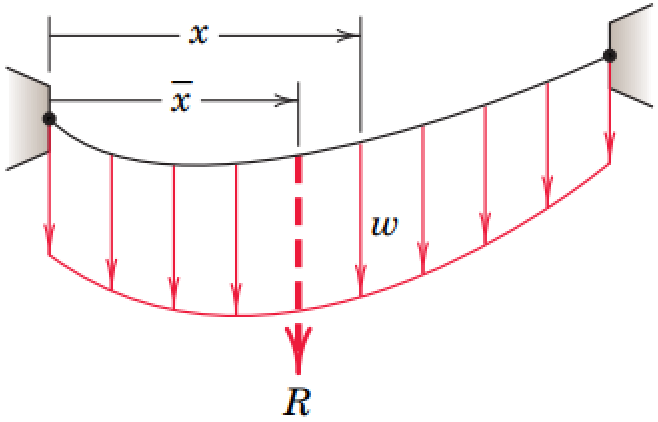

- In some instances the weight of the cable is negligible compared with the loads it supports. In other cases the weight of the cable may be an appreciable load or the sole load and cannot be neglected.

# 5.8 Flexible Cables

- General Relationships
  - Resultant (R) of the variable and continuous load

$$R = \int dR = \int w\, dx$$

$$R\bar{x} = \int x\, dR \qquad \bar{x} = \frac{\int x\, dR}{R}$$

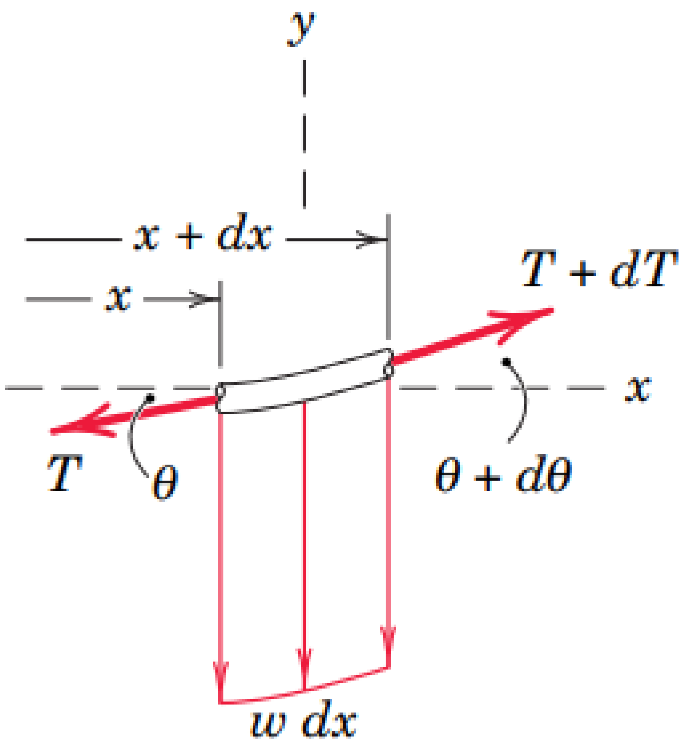


# 5.8 FLEXIBLE CABLES

- The equilibrium condition of the cable is satisfied if each infinitesimal element of the cable is in equilibrium.

$$(T + dT)\sin(\theta + d\theta) = T\sin\theta + w\,dx$$

$$(T + dT)\cos(\theta + d\theta) = T\cos\theta$$




- The trigonometric expansion for the sine and cosine

$$(T + dT)(\sin\theta + \cos\theta\,d\theta) = T\sin\theta + w\,dx$$

$$(T + dT)(\cos\theta - \sin\theta\,d\theta) = T\cos\theta$$

## 5.8 Flexible Cables

- Dropping the second-order terms and simplifying

$$T\cos\theta\, d\theta + dT\sin\theta = w\,dx \quad\rightarrow\quad d(T\sin\theta) = w\,dx$$

$$-T\sin\theta\, d\theta + dT\cos\theta = 0 \quad\rightarrow\quad d(T\cos\theta) = 0$$

the horizontal component of T remains unchanged

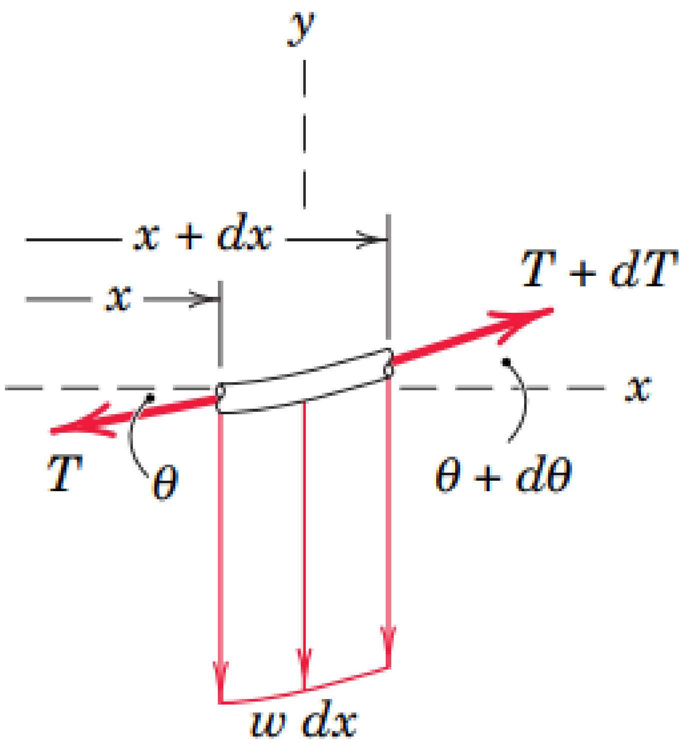
$$T = T_0/\cos\theta$$

$$d(T_0\tan\theta) = w\,dx$$

$\tan\theta = dy/dx$

$$\frac{d^2y}{dx^2} = \frac{w}{T_0}$$

the differential equation for the flexible cable



## 5.8 Flexible Cables

- The differential equation for the flexible cable

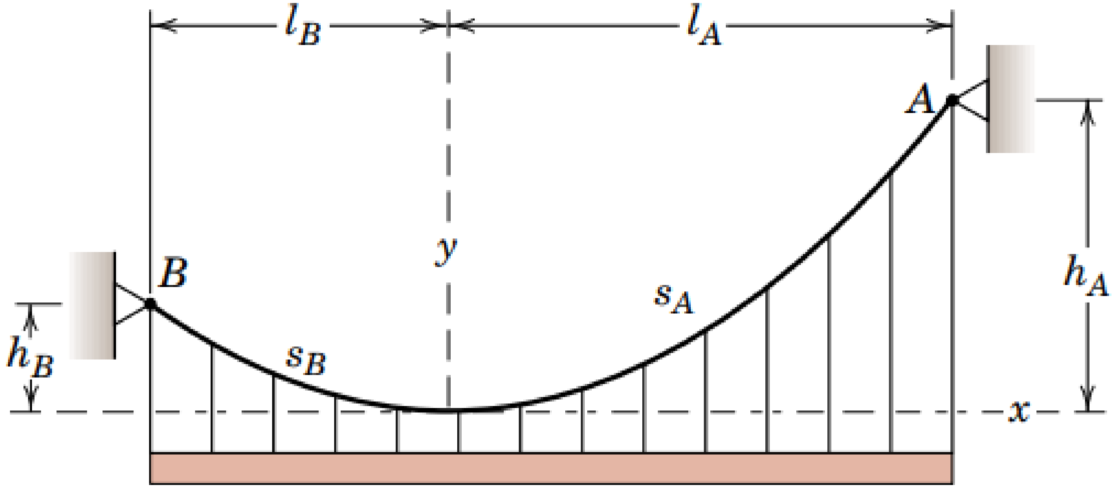
$$\frac{d^2y}{dx^2} = \frac{w}{T_0}$$

- The solution to the equation is that functional relation $y=f(x)$ which satisfies the equation and also satisfies the conditions at the fixed ends of the cable, called boundary conditions

# 5.8 FLEXIBLE CABLES

- Parabolic Cable
  - When w is constant

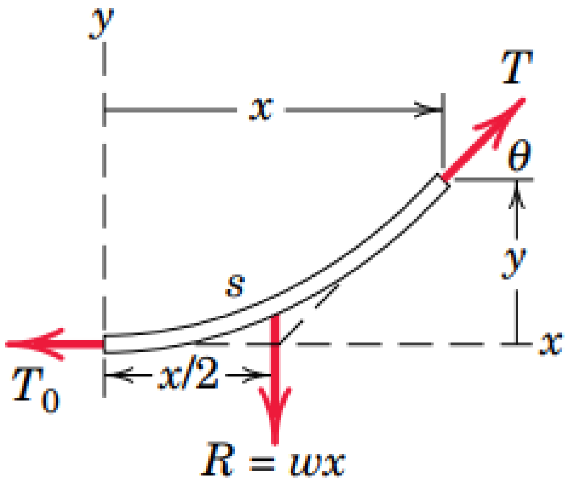
$$\frac{dy}{dx} = \frac{wx}{T_0} + C$$




  - Origin at the lowest point of the cable:

$dy/dx = 0$ when $x = 0$ → $$\frac{dy}{dx} = \frac{wx}{T_0}$$

→ $$\int_0^y dy = \int_0^x \frac{wx}{T_0}\,dx$$ → $$y = \frac{wx^2}{2T_0}$$



## 5.8 Flexible Cables

- Inserting boundary condition

$$y = \frac{wx^2}{2T_0} \xrightarrow{x = l_A \text{ and } y = h_A} T_0 = \frac{wl_A{}^2}{2h_A} \rightarrow y = h_A(x/l_A)^2$$

- From the Pythagorean theorem

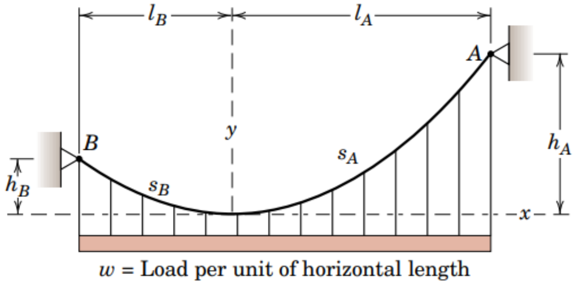
$$T = \sqrt{T_0{}^2 + w^2x^2} = w\sqrt{x^2 + (l_A{}^2/2h_A)^2}$$

- The maximum tension:

$$x = l_A \rightarrow T_{max} = wl_A\sqrt{1 + (l_A/2h_A)^2}$$

# 5.8 FLEXIBLE CABLES

- We obtain the length $s_A$ of the cable from the origin to point A




$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$\rightarrow \int_0^{s_A} ds = \int_0^{l_A} \sqrt{1 + (dy/dx)^2}\, dx = \int_0^{l_A} \sqrt{1 + (wx/T_0)^2}\, dx$$

- Using the binomial expansion $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

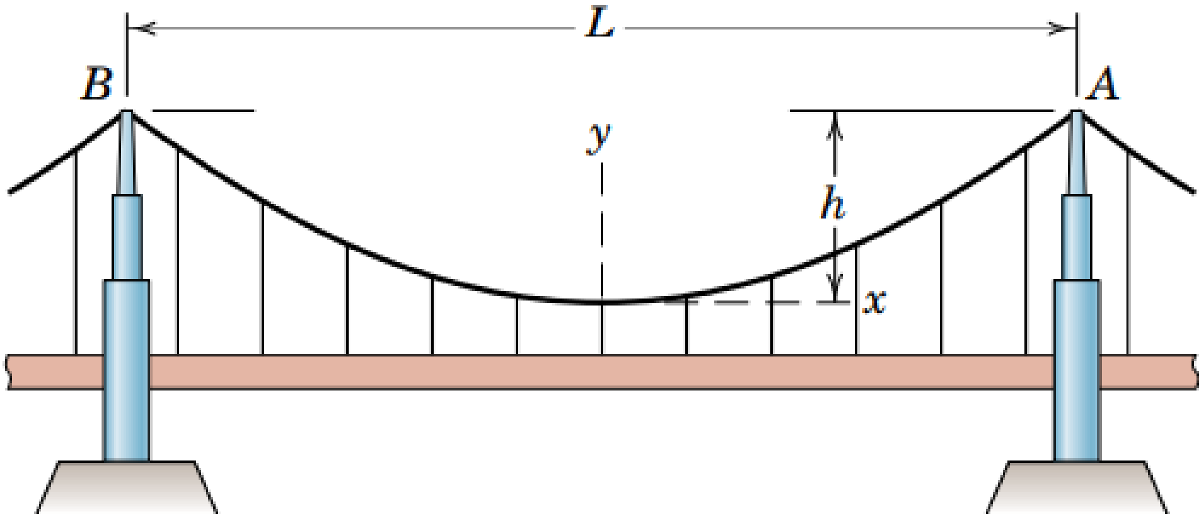
$$\rightarrow s_A = \int_0^{l_A} \left(1 + \frac{w^2x^2}{2T_0^{\,2}} - \frac{w^4x^4}{8T_0^{\,4}} + \cdots\right) dx = l_A\left[1 + \frac{2}{3}\left(\frac{h_A}{l_A}\right)^2 - \frac{2}{5}\left(\frac{h_A}{l_A}\right)^4 + \cdots\right]$$

## 5.8 FLEXIBLE CABLES

- For a suspension bridge

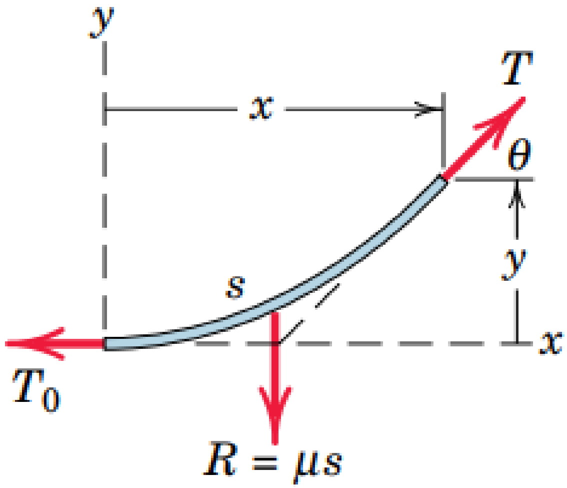
$$T_{\max} = \frac{wL}{2}\sqrt{1 + (L/4h)^2}$$

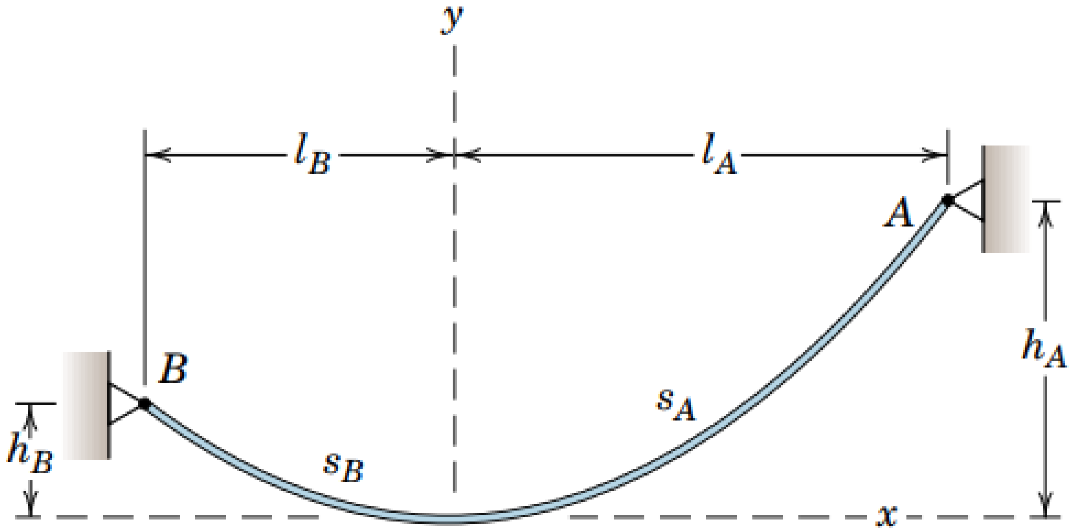
$$S = L\left[1 + \frac{8}{3}\left(\frac{h}{L}\right)^2 - \frac{32}{5}\left(\frac{h}{L}\right)^4 + \cdots\right]$$



# 5.8 FLEXIBLE CABLES

- Catenary Cable
  - A uniform cable, suspended from two points A and B and hanging under the action of its own weight only



# 5.8 FLEXIBLE CABLES

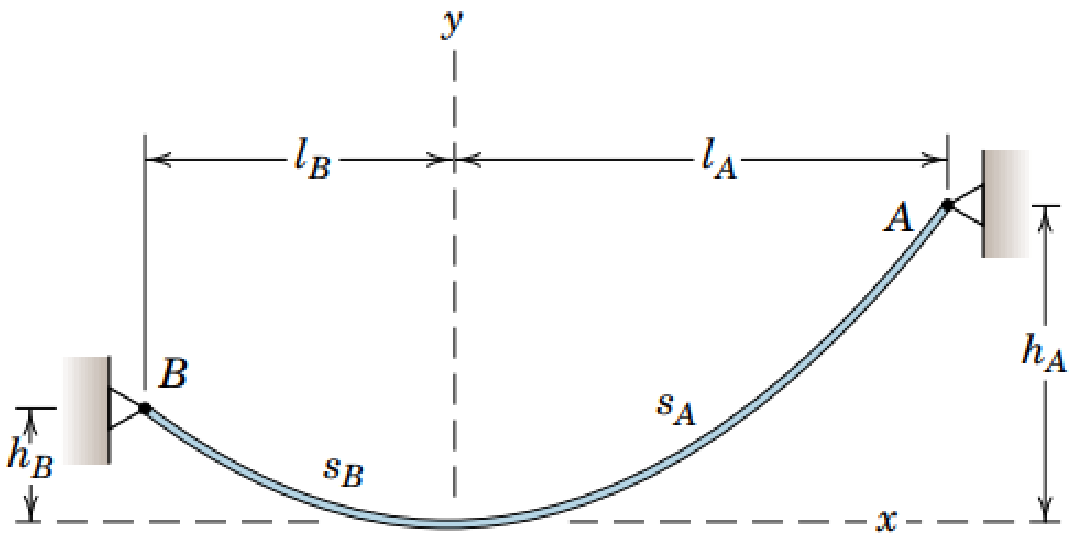
- Catenary Cable

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\frac{ds}{dx}$$

$$y = \frac{T_0}{\mu}\left(\cosh\frac{\mu x}{T_0} - 1\right)$$

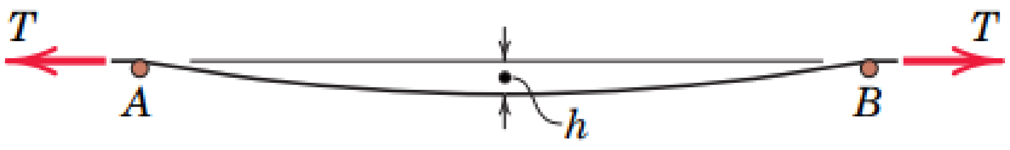
$$s = \frac{T_0}{\mu}\sinh\frac{\mu x}{T_0}$$

$$T = T_0\cosh\frac{\mu x}{T_0} = T_0 + \mu y$$



## Sample Problem 5/16

A 100-ft length of surveyor's tape weighs 0.6 lb. When the tape is stretched between two points on the same level by a tension of 10 lb at each end, calculate the sag $h$ in the middle.




$$\mu = 0.6/100 = 0.006 \text{ lb/ft}$$

$$2s = 100 \text{ or } s = 50 \text{ ft.}$$

$[T^2 = \mu^2 s^2 + T_0^2]$ $\qquad 10^2 = (0.006)^2(50)^2 + T_0^2$

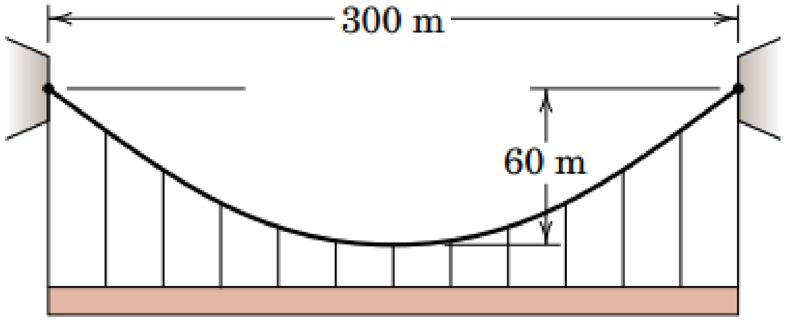
$$T_0 = 9.995 \text{ lb}$$

$[T = T_0 + \mu y]$ $\qquad 10 = 9.995 + 0.006h$

$$h = 0.750 \text{ ft} \quad \text{or} \quad 9.00 \text{ in.}$$

## Sample Problem 5/17

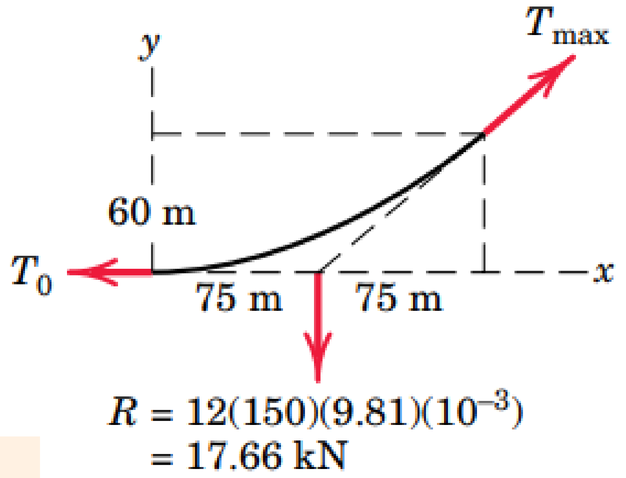
The light cable supports a mass of 12 kg per meter of horizontal length and is suspended between the two points on the same level 300 m apart. If the sag is 60 m, find the tension at midlength, the maximum tension, and the total length of the cable.




$$\left[T_0 = \frac{wL^2}{8h}\right] \qquad T_0 = \frac{0.1177(300)^2}{8(60)} = 22.1 \text{ kN}$$

$$\left[T_{max} = \frac{wL}{2}\sqrt{1 + \left(\frac{L}{4h}\right)^2}\right]$$

$$T_{max} = \frac{12(9.81)(10^{-3})(300)}{2}\sqrt{1 + \left(\frac{300}{4(60)}\right)^2} = 28.3 \text{ kN}$$




$$S = 300\left[1 + \frac{8}{3}\left(\frac{1}{5}\right)^2 - \frac{32}{5}\left(\frac{1}{5}\right)^4 + \cdots\right] = 300[1 + 0.1067 - 0.01024 + \cdots] = 329 \text{ m}$$

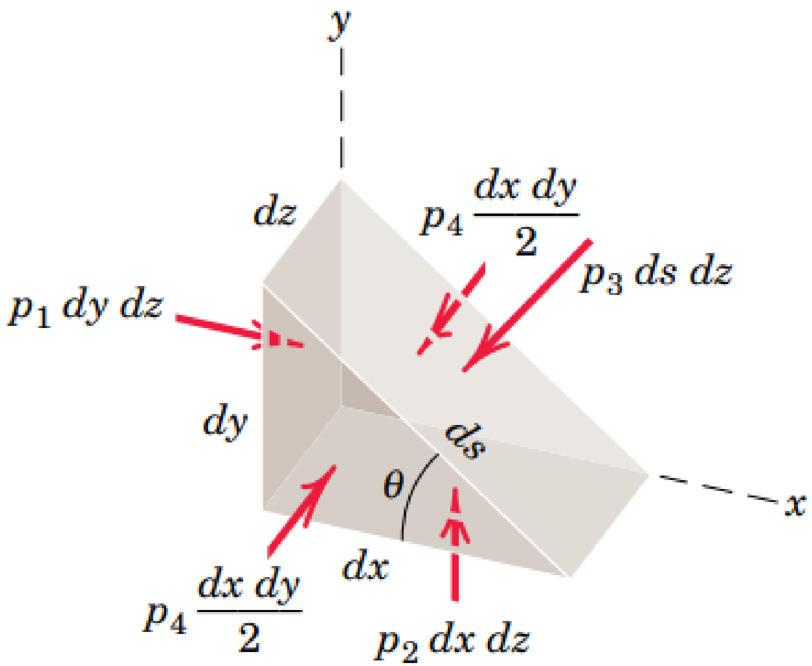
## 5.9 Fluid Statics

- A fluid:
  - Any continuous substance which, when at rest, is unable to support shear force.
  - Thus, a fluid at rest can exert only normal forces on a bounding surface.
  - Fluids may be either gaseous or liquid.
  - The statics of fluids:
    - ✓ "Hydrostatics" when the fluid is a liquid
    - ✓ "Aerostatics" when the fluid is a gas

# 5.9 Fluid Statics

- Fluid Pressure
  - Pascal's law :
    - ✓ Pressure at any given point in a fluid is same in all directions

$$p_1 = p_2 = p_3 = p$$



# 5.9 FLUID STATICS

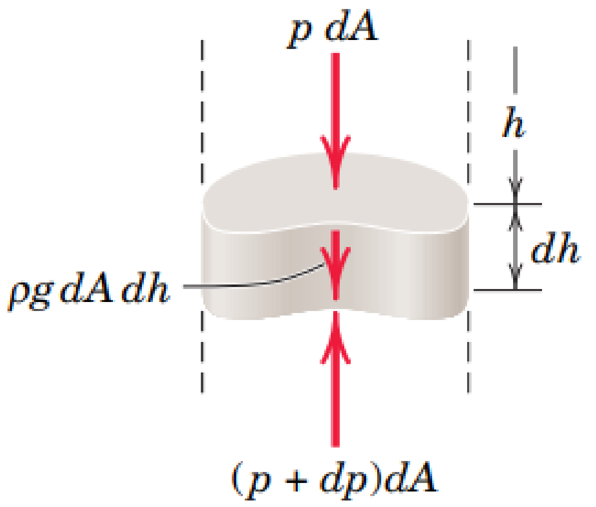
## Fluid Pressure

- In all fluids at rest, the pressure is a function of the vertical dimension

$$p\,dA + \rho g\,dA\,dh - (p + dp)\,dA = 0$$

$$\rightarrow \quad dp = \rho g\,dh$$

$$\rightarrow \quad \boxed{p = p_0 + \rho g h}$$

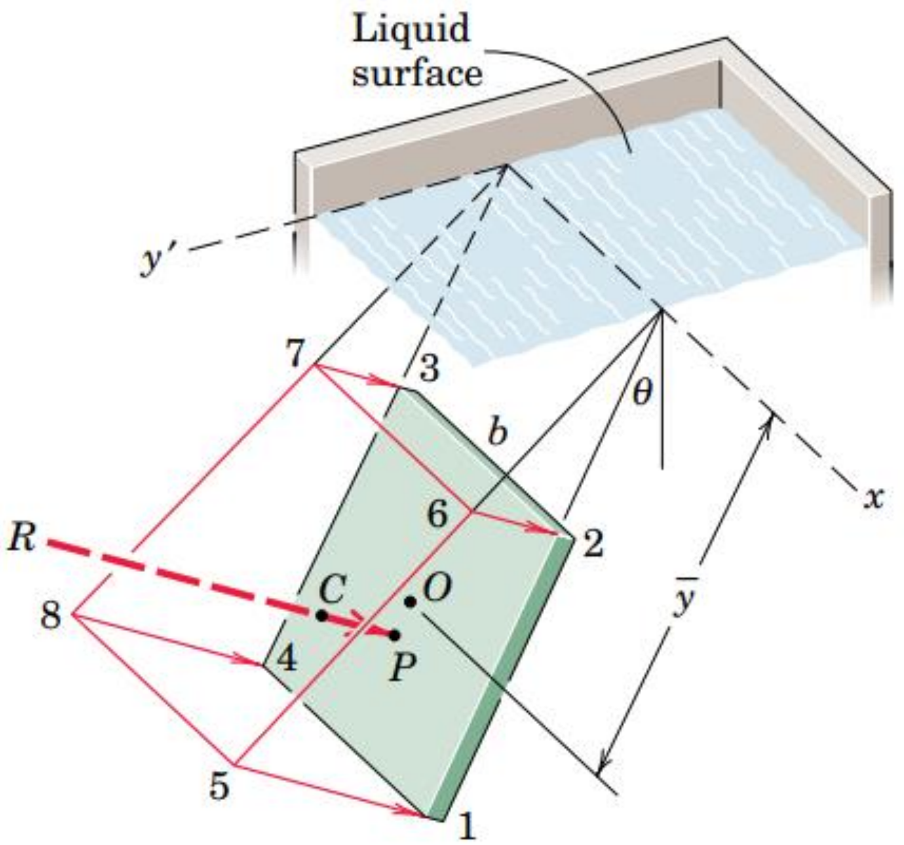



- ✓ The common unit for pressure in SI units is the kilopascal (kPa)

$$p = \rho g h = \left(1.0\,\frac{\text{Mg}}{\text{m}^3}\right)\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(10\ \text{m}) = 98.1\left(10^3\,\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\,\frac{1}{\text{m}^2}\right) = 98.1\ \text{kN/m}^2 = 98.1\ \text{kPa}$$

# 5.9 Fluid Statics

- Hydrostatic Pressure on Submerged Rectangular Surfaces
  - The resultant force acts at some point P called the center of pressure.



# 5.9 Fluid Statics

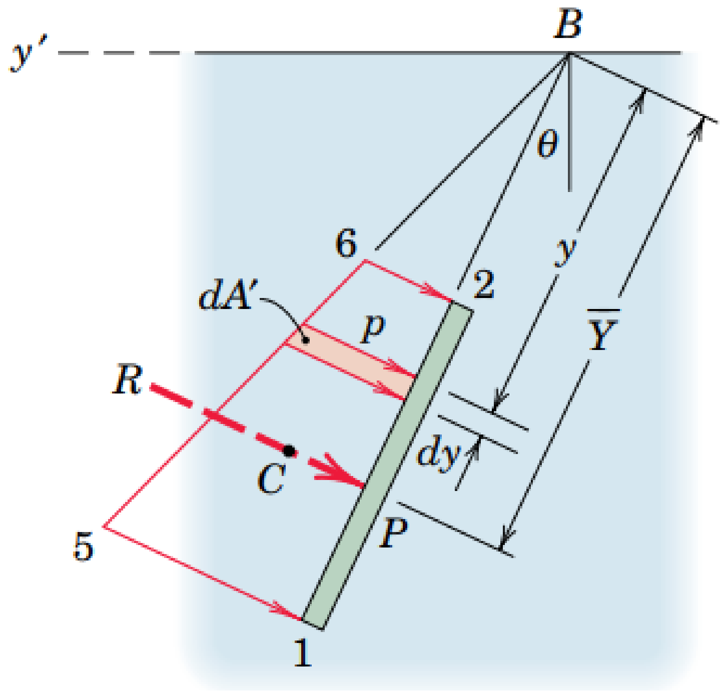
- Hydrostatic Pressure on Submerged Rectangular Surfaces

$$R = b\int dA' = bA'$$

  - R may therefore be written in terms of the average pressure $p_{av}$

$$p_{av} = \frac{1}{2}(p_1 + p_2)$$

$$\rightarrow R = p_{av}A = \rho g\bar{h}A \qquad \bar{h} = \bar{y}\cos\theta$$

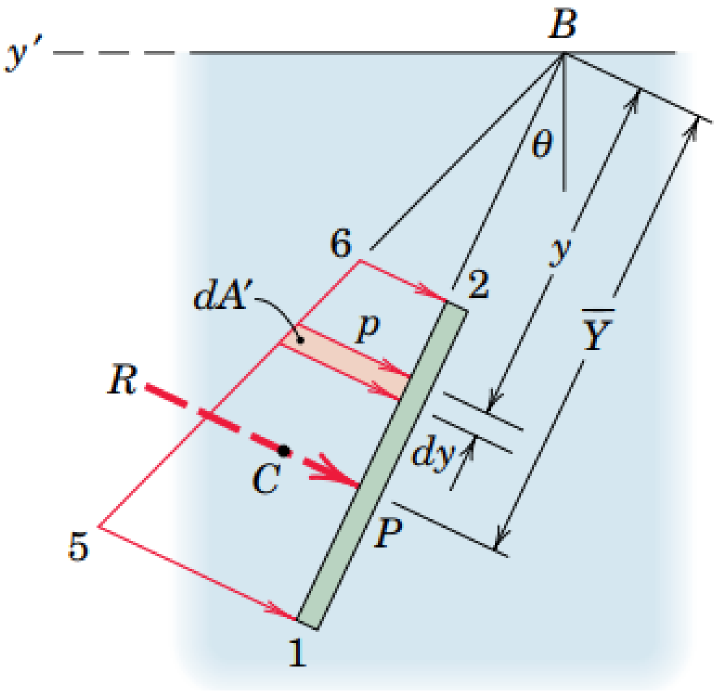


# 5.9 FLUID STATICS

- Hydrostatic Pressure on Submerged Rectangular Surfaces
  - Obtaining the line of action from the principle of moments

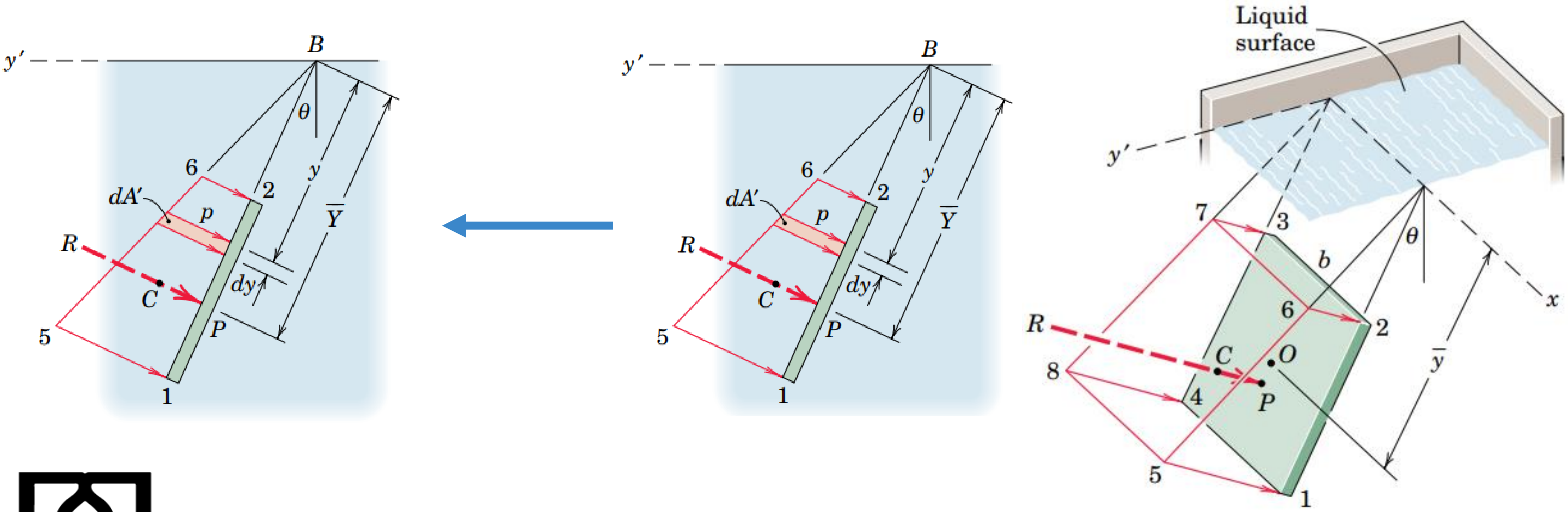
$$\rightarrow \bar{Y} = \frac{\int y \, dA'}{\int dA'}$$

  - R passes through the centroid C of the trapezoidal area defined by the pressure distribution in the vertical section



# 5.9 Fluid Statics

- Hydrostatic Pressure on Submerged Rectangular Surfaces
  - We may simplify the calculation by dividing the trapezoid into a rectangle and a triangle

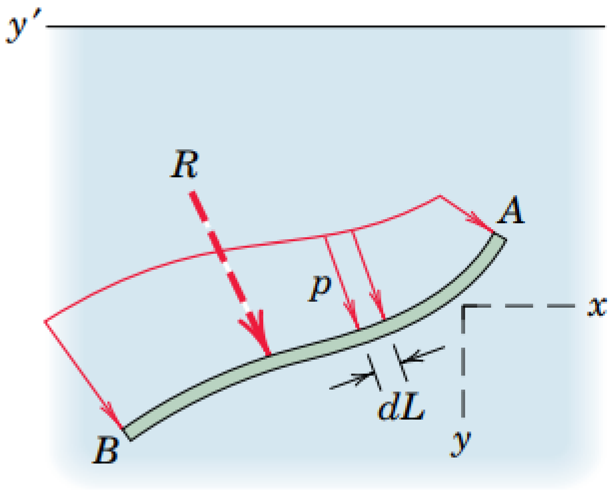


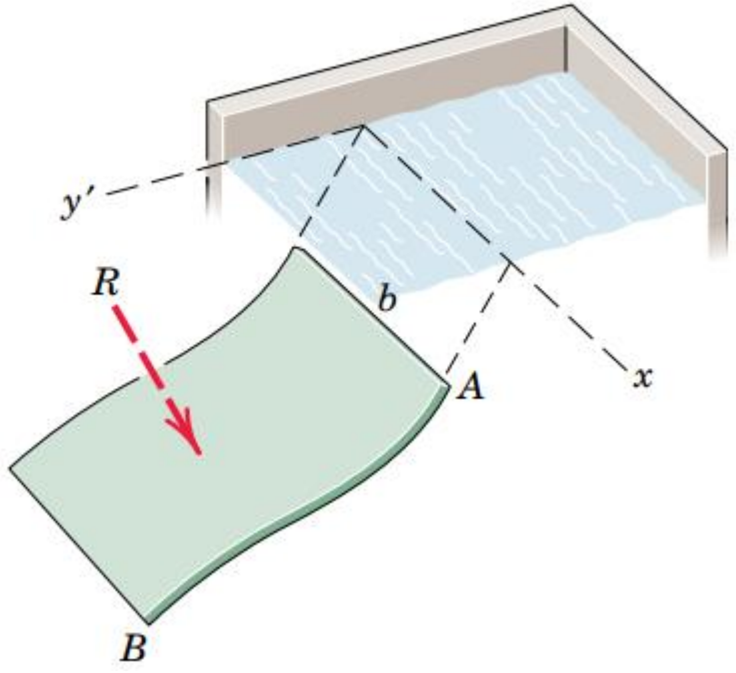
# 5.9 FLUID STATICS

- Hydrostatic Pressure on Cylindrical Surfaces
  - Find R by a direct integration

$$R_x = b\int (p\,dL)_x = b\int p\,dy$$

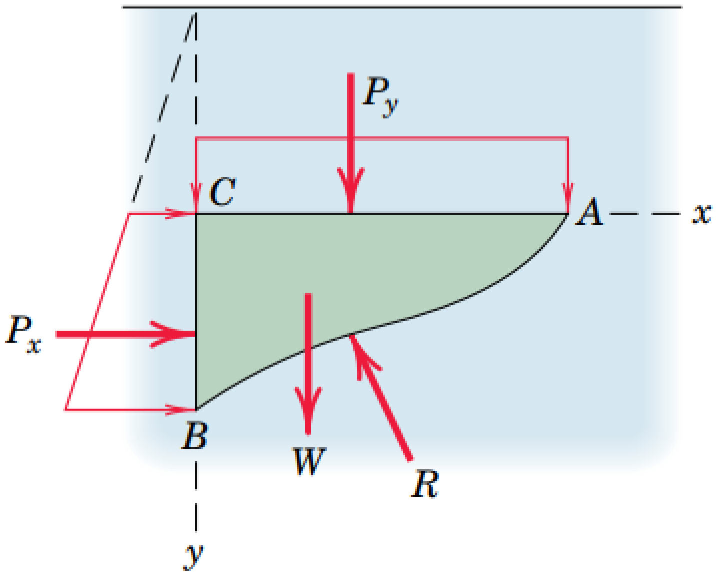
$$R_y = b\int (p\,dL)_y = b\int p\,dx$$

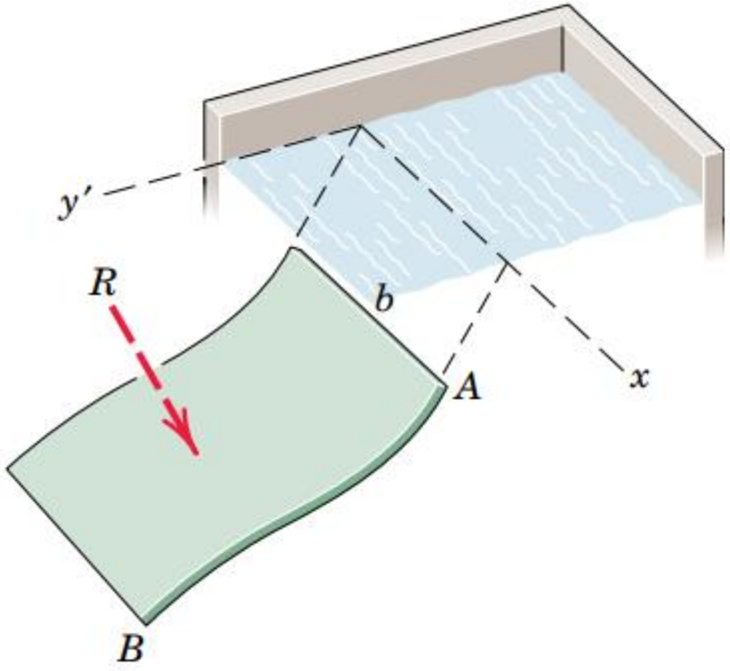


## 5.9 FLUID STATICS

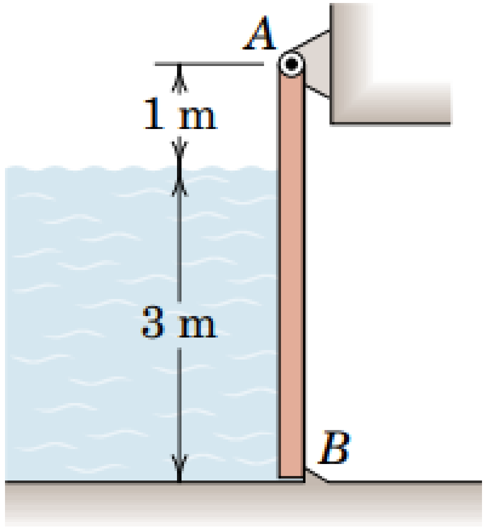
- Hydrostatic Pressure on Cylindrical Surfaces
  - A simpler method: Equilibrium of the block of liquid
    - The equilibrant R is then determined completely from the equilibrium equations which we apply to the free-body diagram of the fluid block.



## Sample Problem 5/19

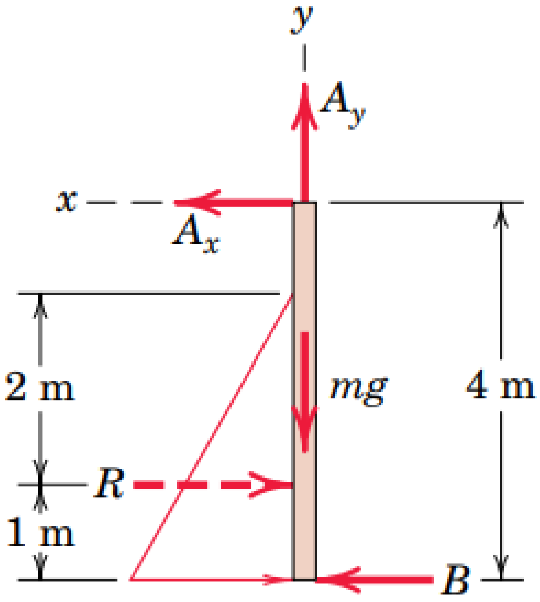
A rectangular plate, shown in vertical section $AB$, is 4 m high and 6 m wide (normal to the plane of the paper) and blocks the end of a fresh-water channel 3 m deep. The plate is hinged about a horizontal axis along its upper edge through $A$ and is restrained from opening by the fixed ridge $B$ which bears horizontally against the lower edge of the plate. Find the force $B$ exerted on the plate by the ridge.




$[p_{av} = \rho g\bar{h}]$ $\qquad p_{av} = 1.000(9.81)(\frac{3}{2}) = 14.72 \text{ kPa}$

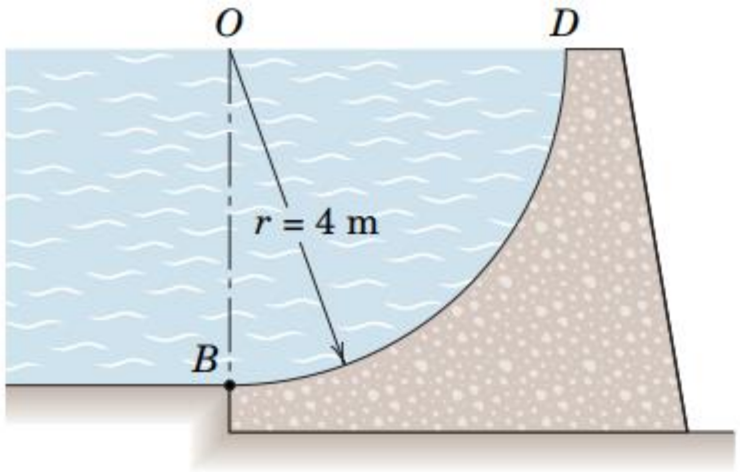
$[R = p_{av} A]$ $\qquad R = (14.72)(3)(6) = 265 \text{ kN}$

$[\Sigma M_A = 0]$ $\qquad 3(265) - 4B = 0 \qquad B = 198.7 \text{ kN}$



## Sample Problem 5/21

Determine completely the resultant force $R$ exerted on the cylindrical dam surface by the water. The density of fresh water is 1.000 Mg/m$^3$, and the dam has a length $b$, normal to the paper, of 30 m.




$$P_x = \rho g \bar{h} A = \frac{\rho g r}{2} br = \frac{(1.000)(9.81)(4)}{2}(30)(4) = 2350 \text{ kN}$$

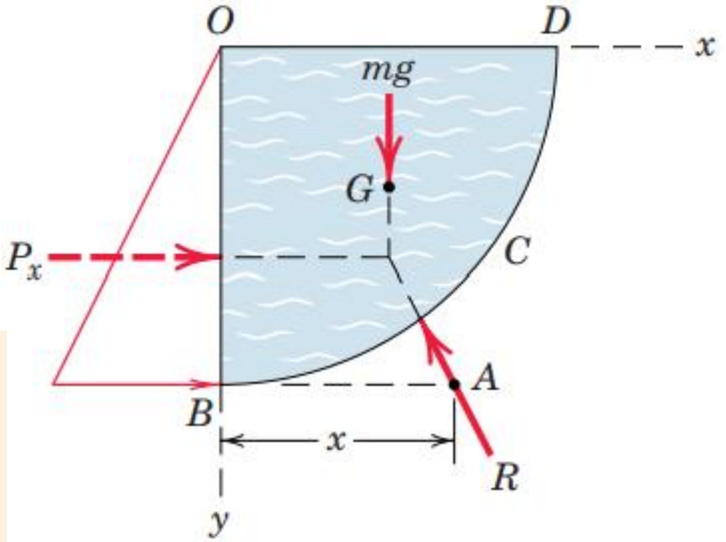
$$mg = \rho g V = (1.000)(9.81)\frac{\pi(4)^2}{4}(30) = 3700 \text{ kN}$$

$$[\Sigma F_x = 0] \qquad R_x = P_x = 2350 \text{ kN}$$

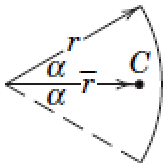
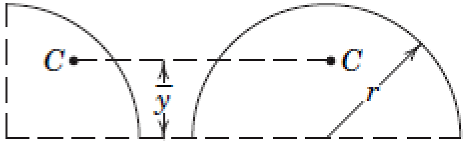
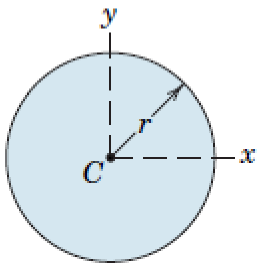
$$[\Sigma F_y = 0] \qquad R_y = mg = 3700 \text{ kN}$$

$$\left[R = \sqrt{R_x^{\,2} + R_y^{\,2}}\right] \qquad R = \sqrt{(2350)^2 + (3700)^2} = 4380 \text{ kN}$$

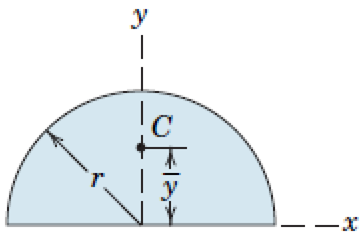
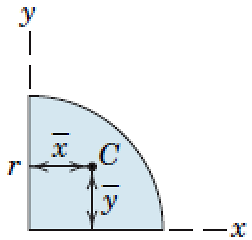
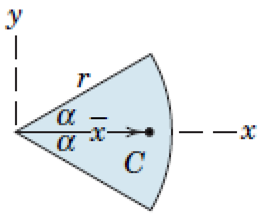
$$P_x \frac{r}{3} + mg\frac{4r}{3\pi} - R_y x = 0, \qquad x = \frac{2350\left(\frac{4}{3}\right) + 3700\left(\frac{16}{3\pi}\right)}{3700} = 2.55 \text{ m}$$



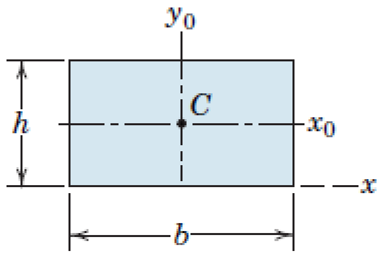
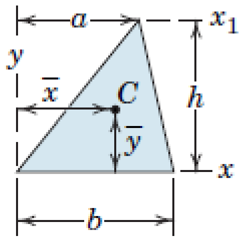
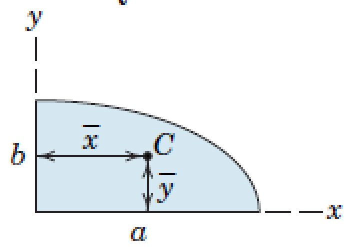
# APPENDIX D

| | | |
|---|---|---|
| Arc Segment   | $\bar{r} = \frac{r \sin\alpha}{\alpha}$ | — |
| Quarter and Semicircular Arcs   | $\bar{y} = \frac{2r}{\pi}$ | — |
| Circular Area   | — | $I_x = I_y = \frac{\pi r^4}{4}$ <br> $I_z = \frac{\pi r^4}{2}$ |

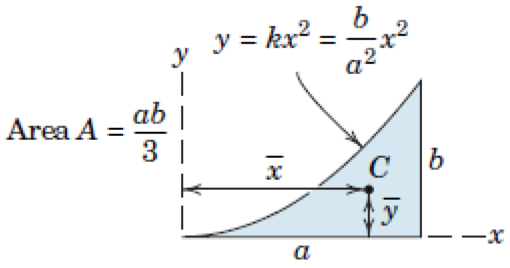
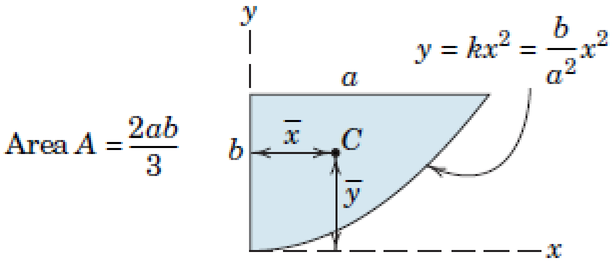
# APPENDIX D

| | | |
|---|---|---|
| Semicircular Area  | $\bar{y} = \frac{4r}{3\pi}$ | $I_x = I_y = \frac{\pi r^4}{8}$<br>$\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$<br>$I_z = \frac{\pi r^4}{4}$ |
| Quarter-Circular Area  | $\bar{x} = \bar{y} = \frac{4r}{3\pi}$ | $I_x = I_y = \frac{\pi r^4}{16}$<br>$\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$<br>$I_z = \frac{\pi r^4}{8}$ |
| Area of Circular Sector  | $\bar{x} = \frac{2}{3}\frac{r \sin\alpha}{\alpha}$ | $I_x = \frac{r^4}{4}(\alpha - \frac{1}{2}\sin 2\alpha)$<br>$I_y = \frac{r^4}{4}(\alpha + \frac{1}{2}\sin 2\alpha)$<br>$I_z = \frac{1}{2} r^4 \alpha$ |

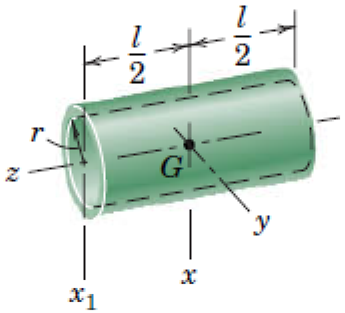
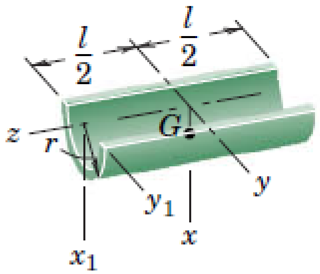
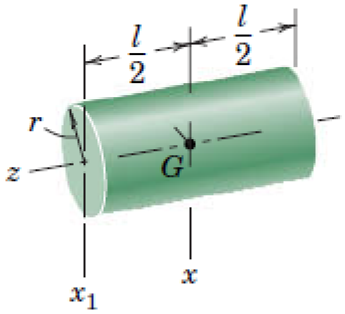
# APPENDIX D

| | | |
|---|---|---|
| Rectangular Area  | — | $I_x = \frac{bh^3}{3}$<br>$\bar{I}_x = \frac{bh^3}{12}$<br>$\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$ |
| Triangular Area  | $\bar{x} = \frac{a+b}{3}$<br>$\bar{y} = \frac{h}{3}$ | $I_x = \frac{bh^3}{12}$<br>$\bar{I}_x = \frac{bh^3}{36}$<br>$I_{x_1} = \frac{bh^3}{4}$ |
| Area of Elliptical Quadrant  | $\bar{x} = \frac{4a}{3\pi}$<br>$\bar{y} = \frac{4b}{3\pi}$ | $I_x = \frac{\pi ab^3}{16}, \quad \bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)ab^3$<br>$I_y = \frac{\pi a^3 b}{16}, \quad \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right)a^3 b$<br>$I_z = \frac{\pi ab}{16}(a^2 + b^2)$ |

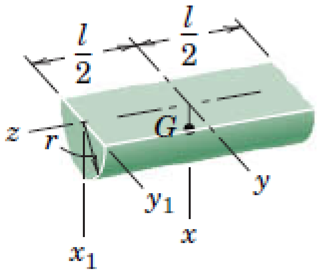
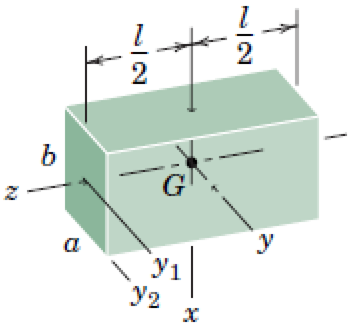
# Appendix D

| | | |
|---|---|---|
| Subparabolic Area   | $\bar{x} = \frac{3a}{4}$<br>$\bar{y} = \frac{3b}{10}$ | $I_x = \frac{ab^3}{21}$<br>$I_y = \frac{a^3b}{5}$<br>$I_z = ab\left(\frac{a^3}{5} + \frac{b^2}{21}\right)$ |
| Parabolic Area   | $\bar{x} = \frac{3a}{8}$<br>$\bar{y} = \frac{3b}{5}$ | $I_x = \frac{2ab^3}{7}$<br>$I_y = \frac{2a^3b}{15}$<br>$I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$ |

# APPENDIX D

| | | | |
|---|---|---|---|
|  | Circular Cylindrical Shell | — | $I_{xx} = \frac{1}{2}mr^2 + \frac{1}{12}ml^2$<br>$I_{x_1x_1} = \frac{1}{2}mr^2 + \frac{1}{3}ml^2$<br>$I_{zz} = mr^2$ |
|  | Half Cylindrical Shell | $\bar{x} = \frac{2r}{\pi}$ | $I_{xx} = I_{yy}$<br>$= \frac{1}{2}mr^2 + \frac{1}{12}ml^2$<br>$I_{x_1x_1} = I_{y_1y_1}$<br>$= \frac{1}{2}mr^2 + \frac{1}{3}ml^2$<br>$I_{zz} = mr^2$<br>$\bar{I}_{zz} = \left(1 - \frac{4}{\pi^2}\right)mr^2$ |
|  | Circular Cylinder | — | $I_{xx} = \frac{1}{4}mr^2 + \frac{1}{12}ml^2$<br>$I_{x_1x_1} = \frac{1}{4}mr^2 + \frac{1}{3}ml^2$<br>$I_{zz} = \frac{1}{2}mr^2$ |

# Appendix D

| | | |
|---|---|---|
|   Semicylinder | $\bar{x} = \frac{4r}{3\pi}$ | $I_{xx} = I_{yy}$<br>$= \frac{1}{4}mr^2 + \frac{1}{12}ml^2$<br>$I_{x_1x_1} = I_{y_1y_1}$<br>$= \frac{1}{4}mr^2 + \frac{1}{3}ml^2$<br>$I_{zz} = \frac{1}{2}mr^2$<br>$\bar{I}_{zz} = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)mr^2$ |
|   Rectangular Parallelepiped | — | $I_{xx} = \frac{1}{12}m(a^2 + l^2)$<br>$I_{yy} = \frac{1}{12}m(b^2 + l^2)$<br>$I_{zz} = \frac{1}{12}m(a^2 + b^2)$<br>$I_{y_1y_1} = \frac{1}{12}mb^2 + \frac{1}{3}ml^2$<br>$I_{y_2y_2} = \frac{1}{3}m(b^2 + l^2)$ |

# APPENDIX D

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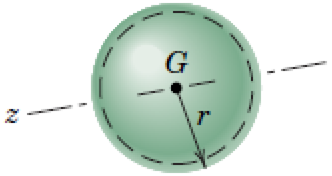
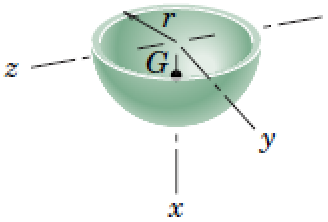
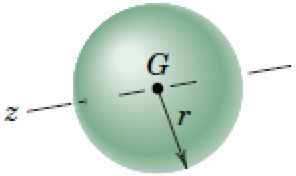
|---|---|---|
|   Spherical Shell | — | $I_{zz} = \frac{2}{3}mr^2$ |
|   Hemispherical Shell | $\bar{x} = \frac{r}{2}$ | $I_{xx} = I_{yy} = I_{zz} = \frac{2}{3}mr^2$<br>$\bar{I}_{yy} = \bar{I}_{zz} = \frac{5}{12}mr^2$ |
|   Sphere | — | $I_{zz} = \frac{2}{5}mr^2$ |

# APPENDIX D

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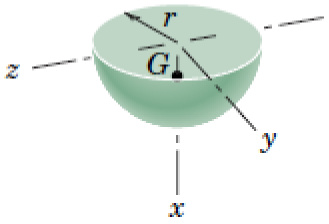
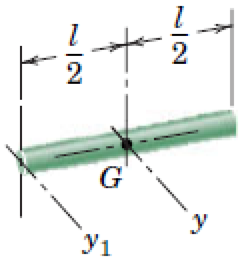
|  | Hemisphere | $\bar{x} = \frac{3r}{8}$ | $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$ <br> $\bar{I}_{yy} = \bar{I}_{zz} = \frac{83}{320}mr^2$ |
|  | Uniform Slender Rod | — | $I_{yy} = \frac{1}{12}ml^2$ <br> $I_{y_1y_1} = \frac{1}{3}ml^2$ |

# APPENDIX D

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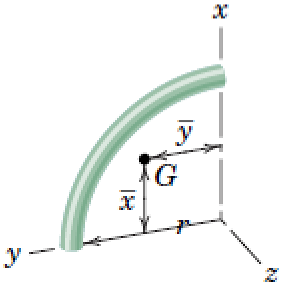
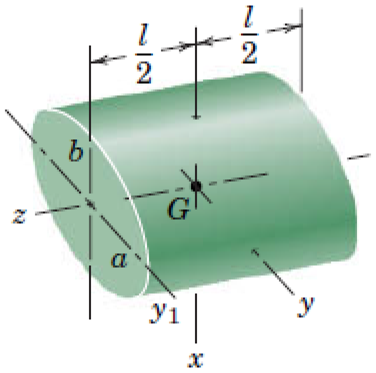
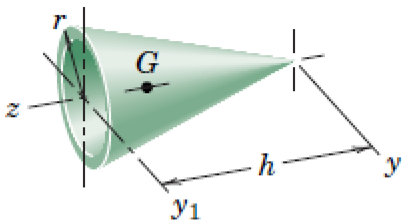
|   Quarter-Circular Rod | $\bar{x} = \bar{y} = \frac{2r}{\pi}$ | $I_{xx} = I_{yy} = \frac{1}{2}mr^2$<br>$I_{zz} = mr^2$ |
|   Elliptical Cylinder | — | $I_{xx} = \frac{1}{4}ma^2 + \frac{1}{12}ml^2$<br>$I_{yy} = \frac{1}{4}mb^2 + \frac{1}{12}ml^2$<br>$I_{zz} = \frac{1}{4}m(a^2 + b^2)$<br>$I_{y_1y_1} = \frac{1}{4}mb^2 + \frac{1}{3}ml^2$ |
|   Conical Shell | $\bar{z} = \frac{2h}{3}$ | $I_{yy} = \frac{1}{4}mr^2 + \frac{1}{2}mh^2$<br>$I_{y_1y_1} = \frac{1}{4}mr^2 + \frac{1}{6}mh^2$<br>$I_{zz} = \frac{1}{2}mr^2$<br>$\bar{I}_{yy} = \frac{1}{4}mr^2 + \frac{1}{18}mh^2$ |

# Appendix D

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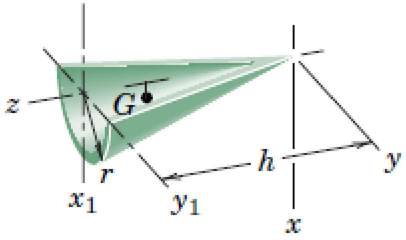
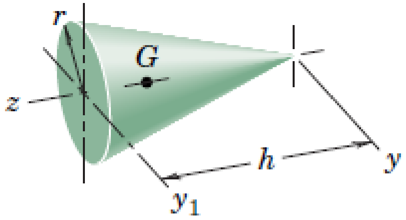
|  | Half Conical Shell | $\bar{x} = \frac{4r}{3\pi}$<br><br>$\bar{z} = \frac{2h}{3}$ | $I_{xx} = I_{yy} = \frac{1}{4}mr^2 + \frac{1}{2}mh^2$<br>$I_{x_1x_1} = I_{y_1y_1} = \frac{1}{4}mr^2 + \frac{1}{6}mh^2$<br>$I_{zz} = \frac{1}{2}mr^2$<br>$\bar{I}_{zz} = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)mr^2$ |
|  | Right Circular Cone | $\bar{z} = \frac{3h}{4}$ | $I_{yy} = \frac{3}{20}mr^2 + \frac{3}{5}mh^2$<br>$I_{y_1y_1} = \frac{3}{20}mr^2 + \frac{1}{10}mh^2$<br>$I_{zz} = \frac{3}{10}mr^2$<br>$\bar{I}_{yy} = \frac{3}{20}mr^2 + \frac{3}{80}mh^2$ |

# APPENDIX D

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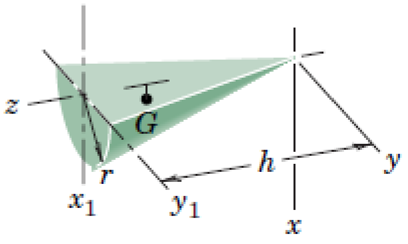
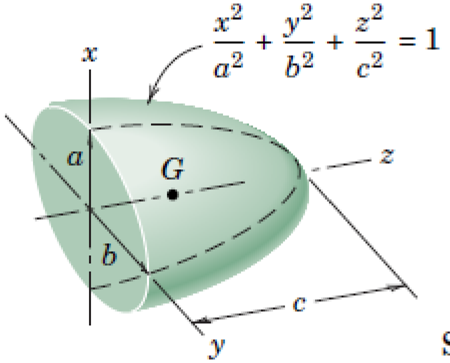
|   Half Cone | $\bar{x} = \frac{r}{\pi}$<br>$\bar{z} = \frac{3h}{4}$ | $I_{xx} = I_{yy} = \frac{3}{20}mr^2 + \frac{3}{5}mh^2$<br>$I_{x_1x_1} = I_{y_1y_1} = \frac{3}{20}mr^2 + \frac{1}{10}mh^2$<br>$I_{zz} = \frac{3}{10}mr^2$<br>$\bar{I}_{zz} = \left(\frac{3}{10} - \frac{1}{\pi^2}\right)mr^2$ |
|   Semiellipsoid | $\bar{z} = \frac{3c}{8}$ | $I_{xx} = \frac{1}{5}m(b^2 + c^2)$<br>$I_{yy} = \frac{1}{5}m(a^2 + c^2)$<br>$I_{zz} = \frac{1}{5}m(a^2 + b^2)$<br>$\bar{I}_{xx} = \frac{1}{5}m(b^2 + \frac{19}{64}c^2)$<br>$\bar{I}_{yy} = \frac{1}{5}m(a^2 + \frac{19}{64}c^2)$ |

# APPENDIX D

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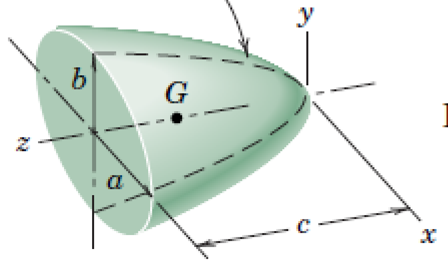
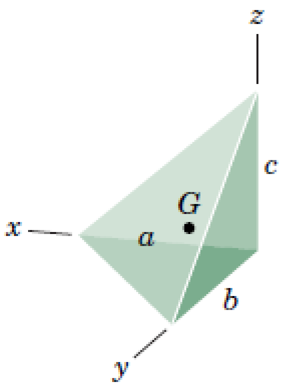
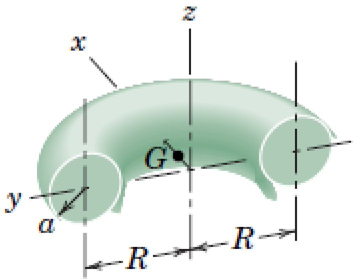
| $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$   | Elliptic Paraboloid | $\bar{z} = \frac{2c}{3}$ | $I_{xx} = \frac{1}{6}mb^2 + \frac{1}{2}mc^2$<br>$I_{yy} = \frac{1}{6}ma^2 + \frac{1}{2}mc^2$<br>$I_{zz} = \frac{1}{6}m(a^2 + b^2)$<br>$\bar{I}_{xx} = \frac{1}{6}m(b^2 + \frac{1}{3}c^2)$<br>$\bar{I}_{yy} = \frac{1}{6}m(a^2 + \frac{1}{3}c^2)$ |
|   | Rectangular Tetrahedron | $\bar{x} = \frac{a}{4}$<br>$\bar{y} = \frac{b}{4}$<br>$\bar{z} = \frac{c}{4}$ | $I_{xx} = \frac{1}{10}m(b^2 + c^2)$<br>$I_{yy} = \frac{1}{10}m(a^2 + c^2)$<br>$I_{zz} = \frac{1}{10}m(a^2 + b^2)$<br>$\bar{I}_{xx} = \frac{3}{80}m(b^2 + c^2)$<br>$\bar{I}_{yy} = \frac{3}{80}m(a^2 + c^2)$<br>$\bar{I}_{zz} = \frac{3}{80}m(a^2 + b^2)$ |
|   | Half Torus | $\bar{x} = \frac{a^2 + 4R^2}{2\pi R}$ | $I_{xx} = I_{yy} = \frac{1}{2}mR^2 + \frac{5}{8}ma^2$<br>$I_{zz} = mR^2 + \frac{3}{4}ma^2$ |