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ROBOTICS

Chapter 3 – Differential Kinematics and Statics

Class Lecture

❑ ***CONTENTS:***

3. Differential Kinematics and Statics

- Differential Kinematics:
  - The relationship between the joint velocities and the corresponding end-effector linear and angular
  - This mapping is described by a matrix, termed *Geometric Jacobian*, which depends on the manipulator configuration.

- Analytical Jacobian:
  - The end-effector pose is expressed with reference to a minimal representation in the operational space, it is possible to compute the Jacobian matrix via differentiation of the direct kinematics function with respect to the joint variables.

3. Differential Kinematics and Statics

- The ***Jacobian*** is used for:
  - Finding singularities
  - Analyzing redundancy
  - Determining inverse kinematics algorithms
  - Describing the mapping between forces applied to the end-effector and resulting torques at the joints (statics)
  - Deriving dynamic equations of motion
  - Designing operational space control schemes

3.1 Geometric Jacobian

- The direct kinematics equation for an n-DOF manipulator:

$$\boldsymbol{T}_e(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}_e(\boldsymbol{q}) & \boldsymbol{p}_e(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{bmatrix} \qquad \boldsymbol{q} = [\, q_1 \quad \dots \quad q_n \,]^T$$

- It is desired to express the end-effector linear velocity $\dot{\boldsymbol{p}}_e$ and angular velocity $\boldsymbol{\omega}_e$ as a function of the joint velocities $\dot{\boldsymbol{q}}$

$$\rightarrow \quad \dot{\boldsymbol{p}}_e = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\rightarrow \quad \boldsymbol{\omega}_e = \boldsymbol{J}_O(\boldsymbol{q})\dot{\boldsymbol{q}}$$

3.1 Geometric Jacobian

- The manipulator differential kinematics equation:

$$v_e = \begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix} = J(q)\dot{q}$$

  - The (6×n) matrix J is the manipulator *geometric Jacobian:*

$$J = \begin{bmatrix} J_P \\ J_O \end{bmatrix}$$

    - ✓ J_P : (3 × n) matrix relating the contribution of joint velocities to end-effector linear velocity
    - ✓ J_O : (3 × n) matrix relating the contribution of joint velocities to end-effector angular velocity

3.1.1 Derivative of a Rotation Matrix

- The derivative of a rotation matrix with respect to time:
  - A time-varying rotation matrix *R(t)*

$$\boldsymbol{R} = \boldsymbol{R}(t)$$

$$\Rightarrow \boldsymbol{R}(t)\boldsymbol{R}^T(t) = \boldsymbol{I}$$

$$\Rightarrow \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) + \boldsymbol{R}(t)\dot{\boldsymbol{R}}^T(t) = \boldsymbol{O}$$

  - Set S:

$$\boldsymbol{S}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t)$$

    - ✓ The (3 × 3) matrix S is skew-symmetric since:

$$\boldsymbol{S}(t) + \boldsymbol{S}^T(t) = \boldsymbol{O}$$

3.1.1 Derivative of a Rotation Matrix

- Postmultiplying both sides by *R(t)* :

$$\boldsymbol{S}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) \quad \longrightarrow \quad \dot{\boldsymbol{R}}(t) = \boldsymbol{S}(t)\boldsymbol{R}(t)$$

  - ✓ The time derivative of *R(t)* is expressed as a function of *R(t)*.

- Consider a constant vector *p'* and the vector *p(t) = R(t) p'*:

$$\dot{\boldsymbol{p}}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{p}' \quad \longrightarrow \quad \dot{\boldsymbol{p}}(t) = \boldsymbol{S}(t)\boldsymbol{R}(t)\boldsymbol{p}'$$

  - ✓ It is known from mechanics that (*ω(t)* denotes the angular velocity of frame R(t) with respect to the reference frame):

$$\longrightarrow \quad \dot{\boldsymbol{p}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{R}(t)\boldsymbol{p}'$$

3.1.1 Derivative of a Rotation Matrix

- Therefore, the matrix operator *S(t)* describes the vector product between the vector ω and the vector *R(t)p'*.

$$\boldsymbol{\omega}(t) = [\omega_x \quad \omega_y \quad \omega_z]^T$$

$$\rightarrow \quad \boldsymbol{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

$$\rightarrow \quad \boldsymbol{S}(t) = \boldsymbol{S}(\boldsymbol{\omega}(t)) \qquad \rightarrow \quad \dot{\boldsymbol{R}} = \boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}$$

- It can be shown that:

$$\boldsymbol{R}\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{R}^T = \boldsymbol{S}(\boldsymbol{R}\boldsymbol{\omega})$$

3.1.1 Derivative of a Rotation Matrix

- Example 3.1
  - The elementary rotation matrix about axis z

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{S}(t) = \begin{bmatrix} -\dot{\alpha}\sin\alpha & -\dot{\alpha}\cos\alpha & 0 \\ \dot{\alpha}\cos\alpha & -\dot{\alpha}\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\dot{\alpha} & 0 \\ \dot{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{S}(\boldsymbol{\omega}(t)).$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & 0 & \dot{\alpha} \end{bmatrix}^T$$

3.1.1 Derivative of a Rotation Matrix

- The coordinate transformation of a point P from Frame 1 to Frame 0:

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

$$\rightarrow \quad \dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \dot{\boldsymbol{R}}_1^0 \boldsymbol{p}^1$$

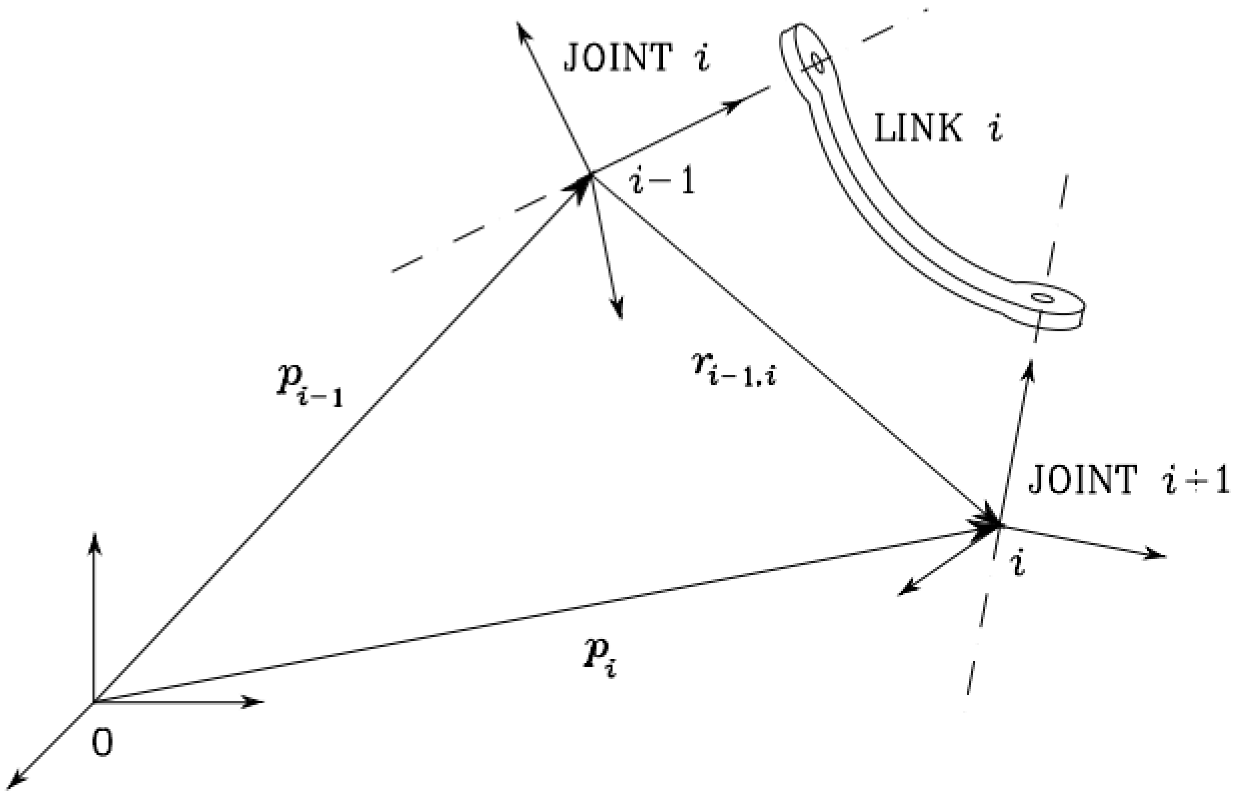
$$\rightarrow \quad \dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{S}(\boldsymbol{\omega}_1^0)\boldsymbol{R}_1^0 \boldsymbol{p}^1$$

$$\rightarrow \quad \dot{\boldsymbol{p}}^0 = \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0$$

  - which is the known form of the velocity composition rule.

3.1.2 Link Velocities

- Using Denavit–Hartenberg convention:



3.1.2 Link Velocities

- p_{i-1} and p_i: Position vectors of the origins of Frames $i-1$ and i

$$\boldsymbol{p}_i = \boldsymbol{p}_{i-1} + \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{R}_{i-1}\dot{\boldsymbol{r}}_{i-1,i}^{i-1} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1}$$

$$= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{i-1,i}$$

- The linear velocity of Link i as a function of the translational and rotational velocities of Link $i-1$

3.1.2 Link Velocities

- Link angular velocity:

$$\boldsymbol{R}_i = \boldsymbol{R}_{i-1}\boldsymbol{R}_i^{i-1}$$

$$\Rightarrow \boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{R}_i = \boldsymbol{S}(\boldsymbol{\omega}_{i-1})\boldsymbol{R}_i + \boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1}$$

$$\Rightarrow \boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1} = \boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_{i-1}^T\boldsymbol{R}_{i-1}\boldsymbol{R}_i^{i-1}$$

$$\Rightarrow \boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1} = \boldsymbol{S}(\boldsymbol{R}_{i-1}\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i$$

$$\Rightarrow \boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{R}_i = \boldsymbol{S}(\boldsymbol{\omega}_{i-1})\boldsymbol{R}_i + \boldsymbol{S}(\boldsymbol{R}_{i-1}\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i$$

$$\Rightarrow \boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \boldsymbol{R}_{i-1}\boldsymbol{\omega}_{i-1,i}^{i-1} = \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i-1,i}$$

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3.1.2 Link Velocities

- Prismatic joint

$$\boldsymbol{\omega}_{i-1,i} = \mathbf{0}$$

$$\boldsymbol{v}_{i-1,i} = \dot{d}_i \boldsymbol{z}_{i-1}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \dot{d}_i \boldsymbol{z}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}$$

3.1.2 Link Velocities

- Revolute joint

$$\boldsymbol{\omega}_{i-1,i} = \dot{\vartheta}_i \boldsymbol{z}_{i-1}$$

$$\boldsymbol{v}_{i-1,i} = \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,i}$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i-1} + \dot{\vartheta}_i \boldsymbol{z}_{i-1}$$

$$\dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}$$

3.1.3 Jacobian Computation

- it is convenient to proceed separately for linear velocity and angular velocity.
  - The *linear velocity*

$$\dot{\boldsymbol{p}}_e = \sum_{i=1}^{n} \frac{\partial \boldsymbol{p}_e}{\partial q_i} \dot{q}_i = \sum_{i=1}^{n} \boldsymbol{\jmath}_{Pi} \dot{q}_i$$

    - ✓ A prismatic joint

$$q_i = d_i \quad \rightarrow \quad \dot{q}_i \boldsymbol{\jmath}_{Pi} = \dot{d}_i \boldsymbol{z}_{i-1} \quad \rightarrow \quad \boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1}$$

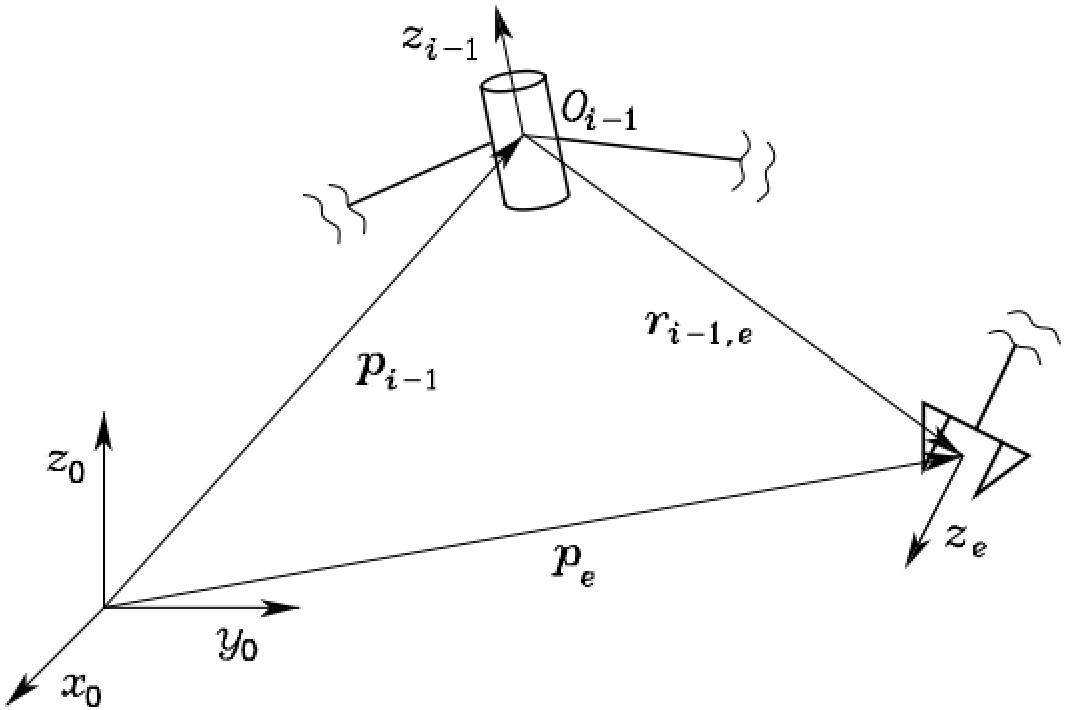
    - ✓ A revolute joint

$$\dot{q}_i \boldsymbol{\jmath}_{Pi} = \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,e} = \dot{\vartheta}_i \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1})$$

$$\rightarrow \quad \boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1})$$

3.1.3 Jacobian Computation

- Velocity contribution of a revolute joint to the end-effector linear velocity



3.1.2 Link Velocities

- The angular velocity

$$\boldsymbol{\omega}_e = \boldsymbol{\omega}_n = \sum_{i=1}^{n} \boldsymbol{\omega}_{i-1,i} = \sum_{i=1}^{n} \boldsymbol{\jmath}_{Oi} \dot{q}_i$$

  - ✓ A prismatic joint

$$\dot{q}_i \boldsymbol{\jmath}_{Oi} = \mathbf{0} \quad \rightarrow \quad \boldsymbol{\jmath}_{Oi} = \mathbf{0}$$

  - ✓ A revolute joint

$$\dot{q}_i \boldsymbol{\jmath}_{Oi} = \dot{\vartheta}_i \boldsymbol{z}_{i-1} \quad \rightarrow \quad \boldsymbol{\jmath}_{Oi} = \boldsymbol{z}_{i-1}$$

3.1.2 Link Velocities

- The Jacobian

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{j}_{P1} & & \boldsymbol{j}_{Pn} \\ & \dots & \\ \boldsymbol{j}_{O1} & & \boldsymbol{j}_{On} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \boldsymbol{j}_{Pi} \\ \boldsymbol{j}_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \boldsymbol{z}_{i-1} \\ \boldsymbol{0} \end{bmatrix} & \text{for a } \textit{prismatic} \text{ joint} \\ \begin{bmatrix} \boldsymbol{z}_{i-1} \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1}) \\ \boldsymbol{z}_{i-1} \end{bmatrix} & \text{for a } \textit{revolute} \text{ joint.} \end{cases}$$

3.1.2 Link Velocities

- The vectors z_{i-1}, p_e and p_{i-1} are all functions of the joint variables:

$$z_{i-1} = R_1^0(q_1)\dots R_{i-1}^{i-2}(q_{i-1})z_0$$

$$\widetilde{p}_e = A_1^0(q_1)\dots A_n^{n-1}(q_n)\widetilde{p}_0$$

$$\widetilde{p}_{i-1} = A_1^0(q_1)\dots$$

$$J^u = \begin{bmatrix} R^u & O \\ O & R^u \end{bmatrix} J$$

$$z_0 = [0 \quad 0 \quad 1]^T$$

$$\widetilde{p}_0 = [0 \quad 0 \quad 0 \quad 1]^T$$

- Jacobian in a different Frame u:

$$\begin{bmatrix} \dot{p}_e^u \\ \omega_e^u \end{bmatrix} = \begin{bmatrix} R^u & O \\ O & R^u \end{bmatrix} \begin{bmatrix} \dot{p}_e \\ \omega_e \end{bmatrix} = \begin{bmatrix} R^u & O \\ O & R^u \end{bmatrix} J\dot{q}$$

$$J^u = \begin{bmatrix} R^u & O \\ O & R^u \end{bmatrix} J$$

3.2 Jacobian of Typical Manipulator Structures

- 3.2.1 Three-link Planar Arm

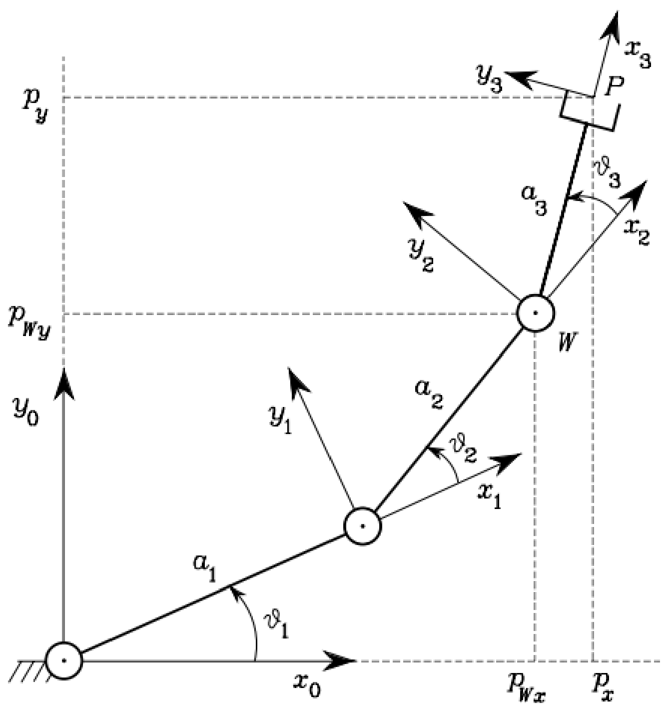
$$J(q) = \begin{bmatrix} z_0 \times (p_3 - p_0) & z_1 \times (p_3 - p_1) & z_2 \times (p_3 - p_2) \\ z_0 & z_1 & z_2 \end{bmatrix}$$

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad p_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix}$$

$$p_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

$$p_3 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix}$$

$$z_0 = z_1 = z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

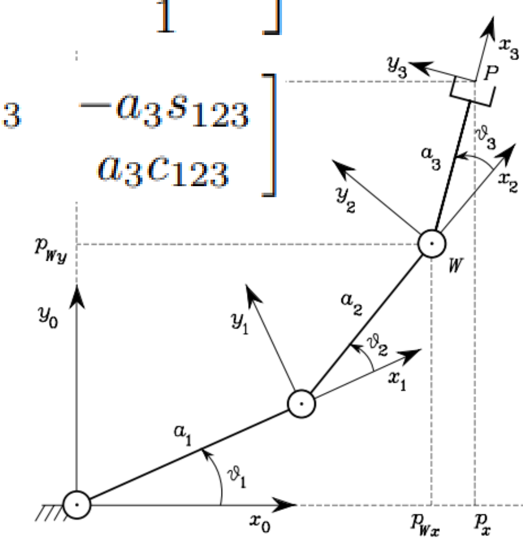


3.2 Jacobian of Typical Manipulator Structures

- 3.2.1 Three-link Planar Arm

$$
\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

$$
\boldsymbol{J}_P = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \end{bmatrix}
$$



3.2 Jacobian of Typical Manipulator Structures

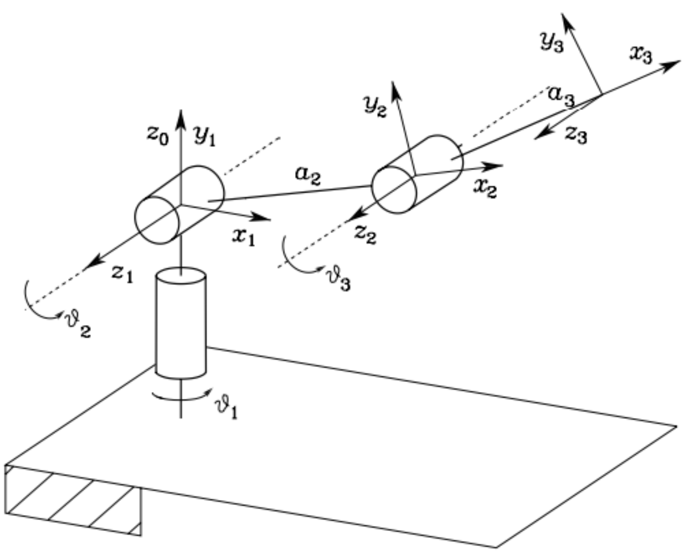
- 3.2.2 Anthropomorphic Arm

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p}_3 - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p}_3 - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p}_3 - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_2 = \begin{bmatrix} a_2 c_1 c_2 \\ a_2 s_1 c_2 \\ a_2 s_2 \end{bmatrix}$$

$$\boldsymbol{p}_3 = \begin{bmatrix} c_1 (a_2 c_2 + a_3 c_{23}) \\ s_1 (a_2 c_2 + a_3 c_{23}) \\ a_2 s_2 + a_3 s_{23} \end{bmatrix}$$

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$



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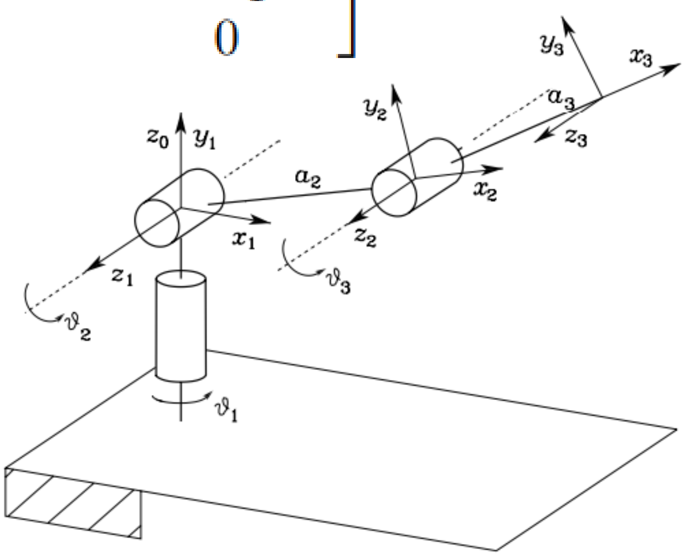
3.2 Jacobian of Typical Manipulator Structures

- 3.2.2 Anthropomorphic Arm

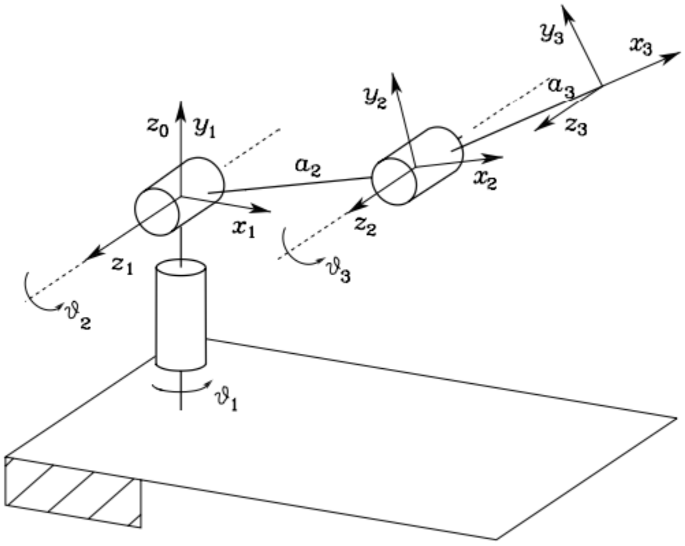
$$
\boldsymbol{J} = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}
$$



3.2 Jacobian of Typical Manipulator Structures

- 3.2.2 Anthropomorphic Arm

$$J_P = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix}$$



3.3 Kinematic Singularities

- The Jacobian in the differential kinematics equation of a manipulator defines a linear mapping: → $\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ $\boldsymbol{v}_e = [\,\dot{\boldsymbol{p}}_e^T \quad \boldsymbol{\omega}_e^T\,]^T$

- ***Kinematic Singularities***: Configurations that J is rank-deficient
  - Singularities represent configurations at which mobility of the structure is reduced, i.e., it is not possible to impose an arbitrary motion to the end-effector.
  - When the structure is at a singularity, infinite solutions to the inverse kinematics problem may exist.
  - In the neighborhood of a singularity, small velocities in the operational space may cause large velocities in the joint space.

3.3 Kinematic Singularities

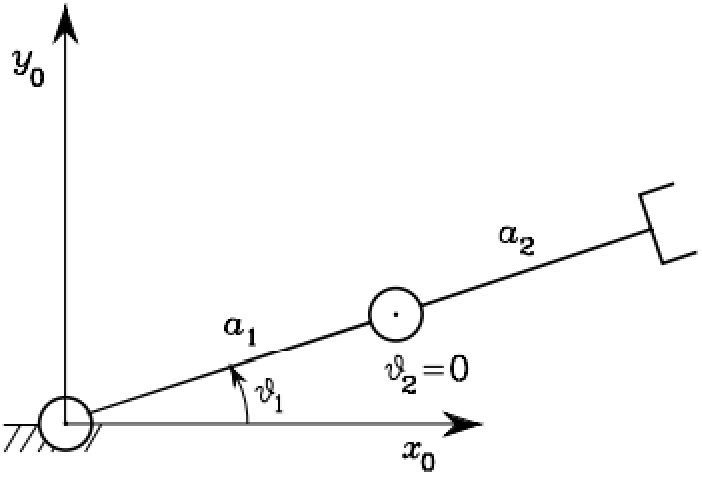
- Boundary singularities:
  - Occur when the manipulator is either outstretched or retracted.
  - Do not represent a true drawback, since they can be avoided on condition that the manipulator is not driven to the boundaries of its reachable workspace.

- Internal singularities:
  - Occur inside the reachable workspace and are generally caused by the alignment of two or more axes of motion, or else by the attainment of particular end-effector configurations.
  - Constitute a serious problem, as they can be encountered anywhere in the reachable workspace for a planned path in the operational space.

3.3 Kinematic Singularities

- Example 3.2
  - A two-link planar arm

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$



$$\rightarrow \quad \det(J) = a_1 a_2 s_2$$

$$\rightarrow \quad \vartheta_2 = 0 \qquad \vartheta_2 = \pi$$

  - When the arm tip is located either on the outer ($\vartheta 2 = 0$) or on the inner ($\vartheta 2 = \pi$) boundary of the reachable workspace.
  - Two column vectors of the Jacobian become parallel, and thus the Jacobian rank becomes one.

3.6 Analytical Jacobian

- If the end-effector pose is specified in terms of a minimal number of parameters, it is possible to compute the Jacobian via differentiation of the direct kinematics function with respect to the joint variables.

- The *Analytical Technique:*

$$\rightarrow \quad \dot{\boldsymbol{p}}_e = \frac{\partial \boldsymbol{p}_e}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}_P(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

$$\rightarrow \quad \dot{\boldsymbol{\phi}}_e = \frac{\partial \boldsymbol{\phi}_e}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}_\phi(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \dot{\boldsymbol{\phi}}_e \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_P(\boldsymbol{q}) \\ \boldsymbol{J}_\phi(\boldsymbol{q}) \end{bmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}_A(\boldsymbol{q}) \dot{\boldsymbol{q}} \quad \rightarrow \quad \boldsymbol{J}_A(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}(\boldsymbol{q})}{\partial \boldsymbol{q}}$$

3.8 Statics

- The goal of statics is to determine the relationship between the generalized forces applied to the end-effector and the generalized forces applied to the joints (forces for prismatic joints, torques for revolute joints) with the manipulator at an equilibrium configuration.

- The application of the *principle of virtual work:*
  - *Joint Torques*

$$dW_{\tau} = \boldsymbol{\tau}^T d\boldsymbol{q}$$

  - End-effector forces

$$dW_{\gamma} = \boldsymbol{f}_e^T d\boldsymbol{p}_e + \boldsymbol{\mu}_e^T \boldsymbol{\omega}_e dt$$

3.8 Statics

- By accounting for the differential kinematics relationship:

$$dW_\gamma = \boldsymbol{f}_e^T d\boldsymbol{p}_e + \boldsymbol{\mu}_e^T \boldsymbol{\omega}_e dt$$

$$\gamma_e = [\,\boldsymbol{f}_e^T \quad \boldsymbol{\mu}_e^T\,]^T \quad \rightarrow \quad \begin{aligned} dW_\gamma &= \boldsymbol{f}_e^T \boldsymbol{J}_P(\boldsymbol{q})d\boldsymbol{q} + \boldsymbol{\mu}_e^T \boldsymbol{J}_O(\boldsymbol{q})d\boldsymbol{q} \\ &= \gamma_e^T \boldsymbol{J}(\boldsymbol{q})d\boldsymbol{q} \end{aligned}$$

$$\rightarrow \quad \begin{aligned} \delta W_\tau &= \boldsymbol{\tau}^T \delta\boldsymbol{q} \\ \delta W_\gamma &= \gamma_e^T \boldsymbol{J}(\boldsymbol{q})\delta\boldsymbol{q} \end{aligned}$$

- The manipulator is at static equilibrium if and only if:

$$\delta W_\tau = \delta W_\gamma \qquad \forall \delta\boldsymbol{q} \quad \rightarrow \quad \boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\gamma_e$$