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Semnan University
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دانشکده مهندسی مکانیک

دانشکده مهندسی مکانیک

درس رباتیک

ROBOTICS

Chapter 2 – Kinematics

Class Lecture

❑ *Contents:*

2. Kinematics

- A manipulator:
  - Kinematic chain of rigid bodies (links) connected by means of revolute or prismatic joints.

- The derivation of the **Direct Kinematics Equation** allows the end-effector position and orientation (pose) to be expressed as a function of the joint variables.

- With reference to a minimal representation of orientation, the concept of **Operational Space** is introduced and its relationship with the **Joint Space** is established.

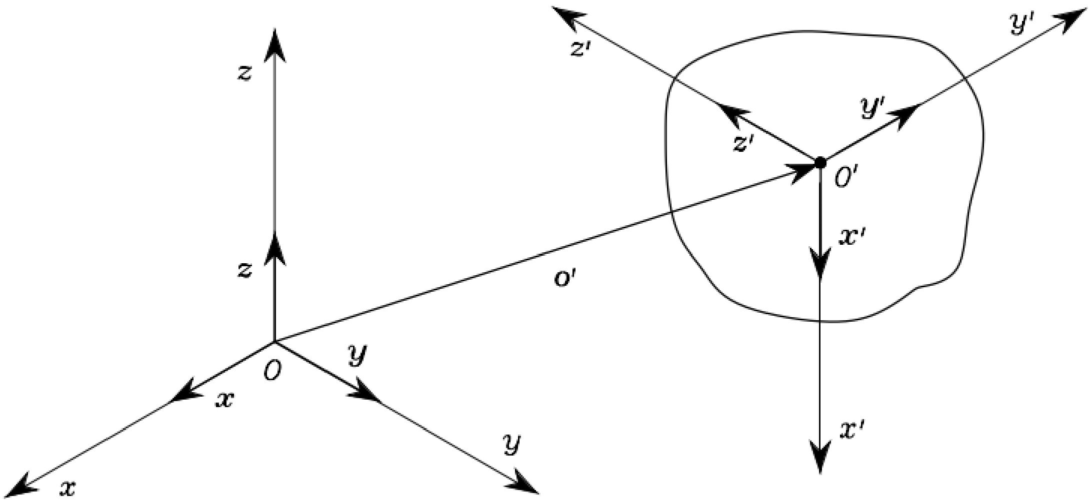
2.1 Pose of a Rigid Body

- A rigid body is completely described in space by its **position** and **orientation** (in brief pose) with respect to a reference frame.

$$\boldsymbol{o}' = o'_x \boldsymbol{x} + o'_y \boldsymbol{y} + o'_z \boldsymbol{z},$$

- Components of the vector along the frame axes

$$\boldsymbol{o}' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}$$



2.1 Pose of a Rigid Body

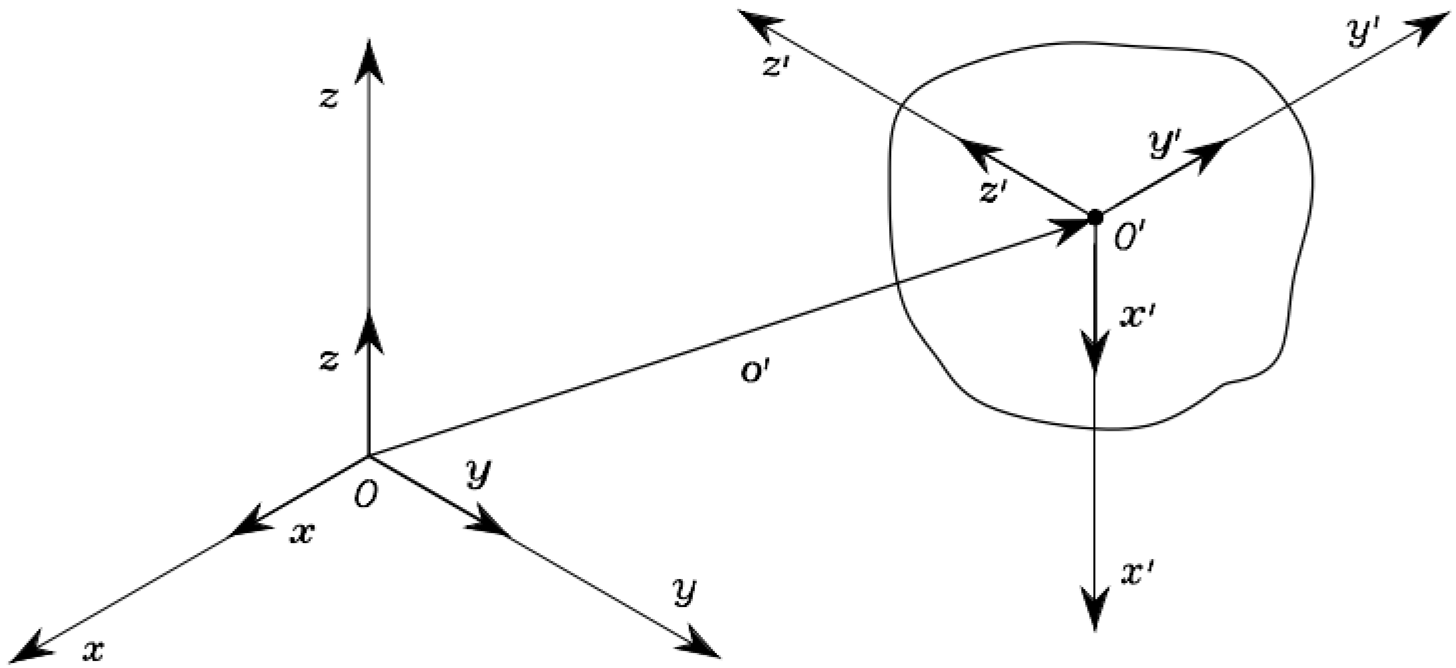
- O–xyz: Reference frame
- O'–x'y'z': Orthonormal frame attached to the body and express its unit vectors with respect to the reference frame.

$$\boldsymbol{x}' = x'_x\boldsymbol{x} + x'_y\boldsymbol{y} + x'_z\boldsymbol{z}$$
$$\boldsymbol{y}' = y'_x\boldsymbol{x} + y'_y\boldsymbol{y} + y'_z\boldsymbol{z}$$
$$\boldsymbol{z}' = z'_x\boldsymbol{x} + z'_y\boldsymbol{y} + z'_z\boldsymbol{z}.$$

  - The components of each unit vector are the direction cosines of the axes of frame O'–x'y'z' with respect to the reference frame O–xyz.

2.2 Rotation Matrix

- O–xyz and O'–x'y'z' frames



2.2 Rotation Matrix

- Unit vectors describing body orientation with respect to reference frame

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

- Column vectors of matrix R are **mutually orthogonal** since they represent the unit vectors of an orthonormal frame

$$\boldsymbol{x}'^T\boldsymbol{y}' = 0 \qquad \boldsymbol{y}'^T\boldsymbol{z}' = 0 \qquad \boldsymbol{z}'^T\boldsymbol{x}' = 0.$$

- Also, they have unit norm

$$\boldsymbol{x}'^T\boldsymbol{x}' = 1 \qquad \boldsymbol{y}'^T\boldsymbol{y}' = 1 \qquad \boldsymbol{z}'^T\boldsymbol{z}' = 1.$$

2.2 Rotation Matrix

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

- As a consequence, R is an orthogonal matrix

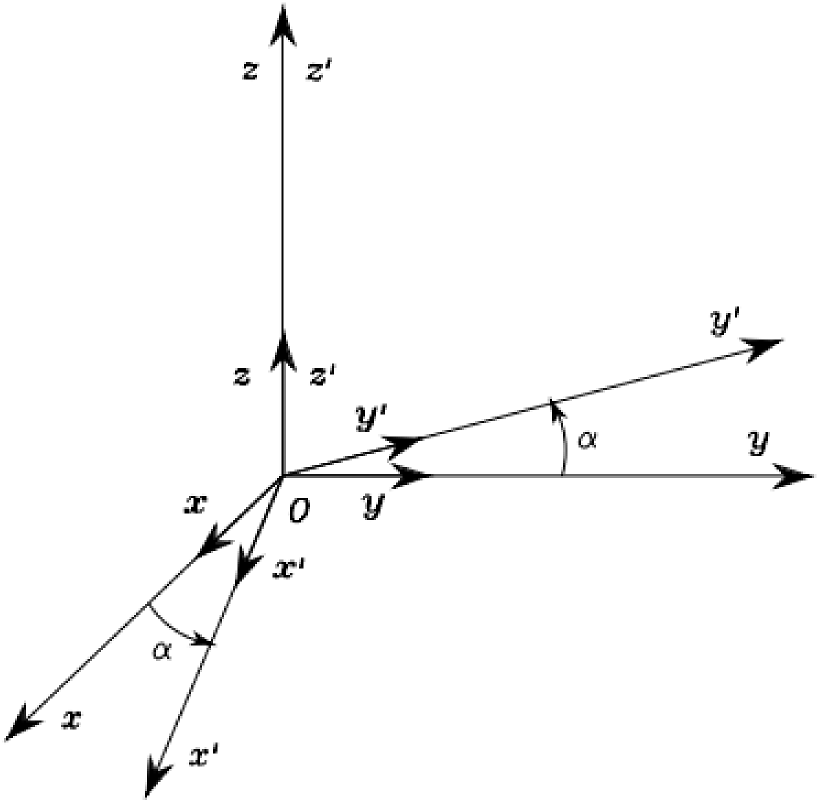
$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_3$$

$$\rightarrow \quad \boldsymbol{R}^T = \boldsymbol{R}^{-1}$$

- Right-handed frame: $det(R) = 1$
- Left-handed frame: $det(R) = -1$

2.2.1 Elementary Rotations

- Elementary rotations of the reference frame about one of the coordinate axes
  - Reference frame O–xyz is rotated by an angle $\boldsymbol{\alpha}$ about axis $\boldsymbol{z}$

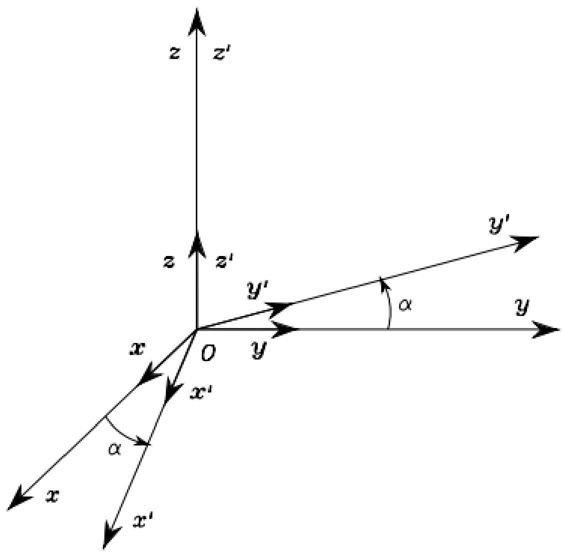


$$x' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \qquad y' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad z' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2.2.1 Elementary Rotations

- Rotation matrix of frame O–x’y’z’ with respect to frame O-xyz is

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



2.2.1 Elementary Rotations

- Rotations by an angle $\boldsymbol{\beta}$ about axis $\boldsymbol{y}$

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

- Rotation by an angle $\boldsymbol{\gamma}$ about axis $\boldsymbol{x}$

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

- Also:

$$\rightarrow \quad \boldsymbol{R}_k(-\vartheta) = \boldsymbol{R}_k^T(\vartheta) \qquad k = x, y, z.$$

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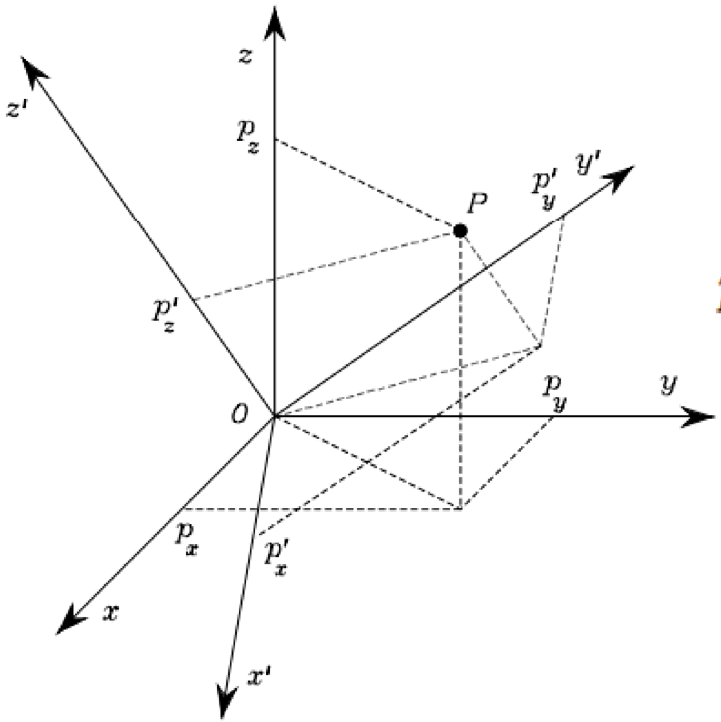
2.2.2 Representation of a Vector

- With coincident origins



$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

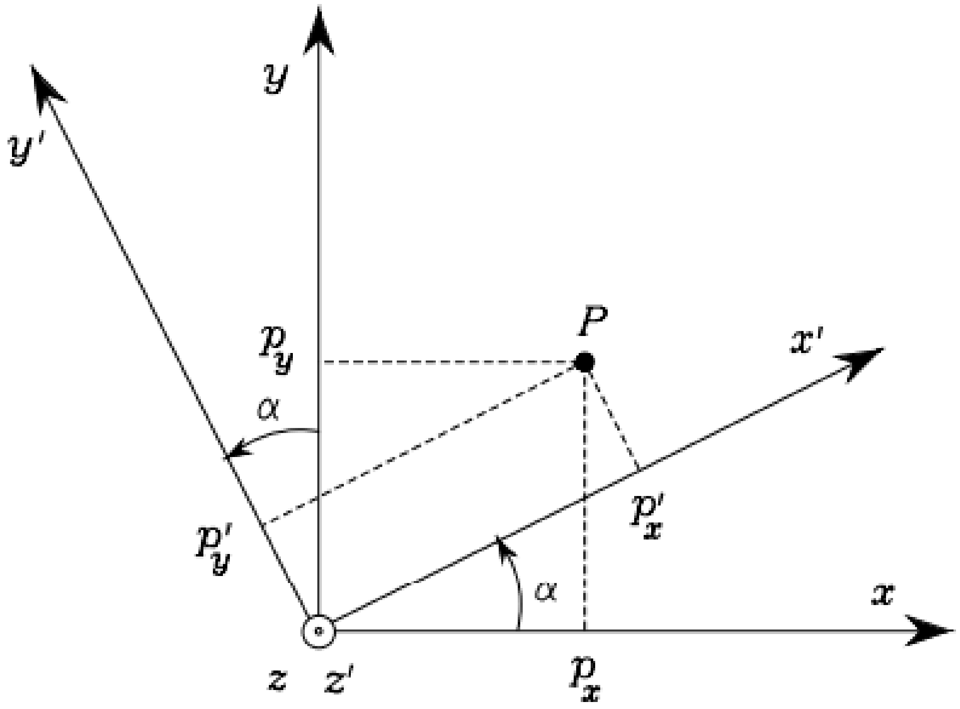
$$\boldsymbol{p} = p'_x \boldsymbol{x}' + p'_y \boldsymbol{y}' + p'_z \boldsymbol{z}' = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}'$$

→ $\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'.$

→ $\boldsymbol{p}' = \boldsymbol{R}^T \boldsymbol{p}.$

2.2.2 Representation of a Vector

- Example 2.1



$$
\begin{aligned}
p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\
p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\
p_z &= p'_z.
\end{aligned}
$$

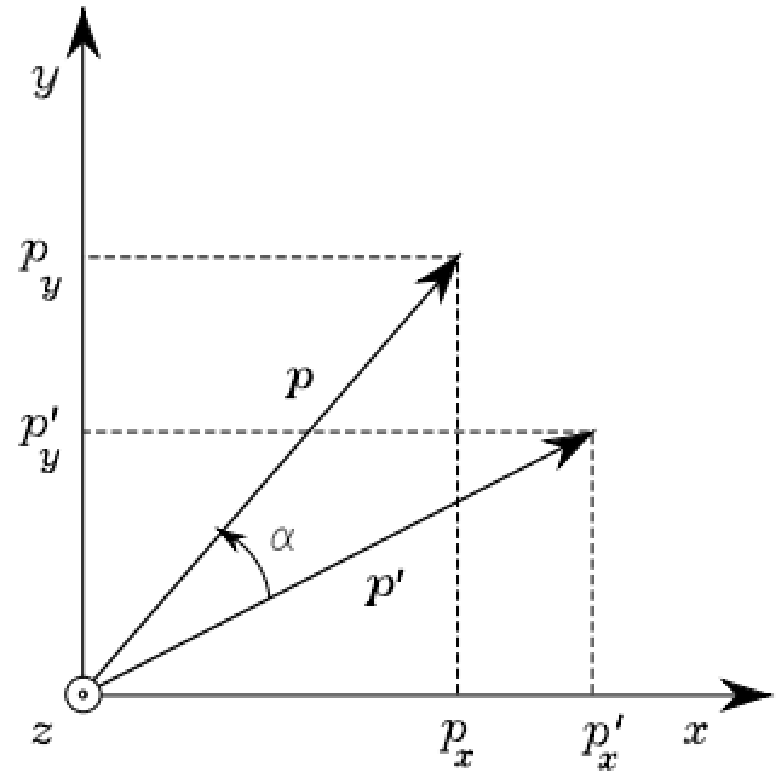
2.2.3 Rotation of a Vector

- A rotation matrix can be also interpreted as the **matrix operator** allowing rotation of a vector by a given angle about an arbitrary axis in space.

- Example 2.2

$$p_x = p'_x \cos\alpha - p'_y \sin\alpha$$
$$p_y = p'_x \sin\alpha + p'_y \cos\alpha$$
$$p_z = p'_z.$$

$$\rightarrow \quad \boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}'$$



2.2.3 Rotation of a Vector

- A rotation matrix attains three equivalent geometrical meanings:

  - Mutual orientation between two coordinate frames

    (its column vectors are the direction cosines of the axes of the rotated frame with respect to the original frame)

  - Coordinate transformation between the coordinates of a point expressed in two different frames (with common origin)

  - An operator that allows the rotation of a vector in the same coordinate frame.

2.3 Composition of Rotation Matrices

- $O\text{–}x_0y_0z_0$, $O\text{–}x_1y_1z_1$, $O\text{–}x_2y_2z_2$ (three frames with common origin O)
- The vector p: position of a generic point in space
  - $\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{p}^2$: the expressions of $\boldsymbol{p}$ in the three frames.

$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1 \boldsymbol{p}^2$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0 \boldsymbol{p}^1$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0 \boldsymbol{p}^2$$

$$\rightarrow \quad \boldsymbol{R}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_2^1$$

  - The overall rotation can be expressed as a sequence of partial rotations

- Also:

$$\rightarrow \quad \boldsymbol{R}_i^j = (\boldsymbol{R}_j^i)^{-1} = (\boldsymbol{R}_j^i)^T$$

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2.3 Composition of Rotation Matrices

- The frame with respect to which the rotation occurs is termed **current frame**.

- Composition of successive rotations is then obtained by **postmultiplication** of the rotation matrices following the given order of rotations

2.3 Composition of Rotation Matrices

- Successive rotations can be also specified by constantly referring them to the **initial frame**.
- In this case, the rotations are made with respect to a fixed frame.

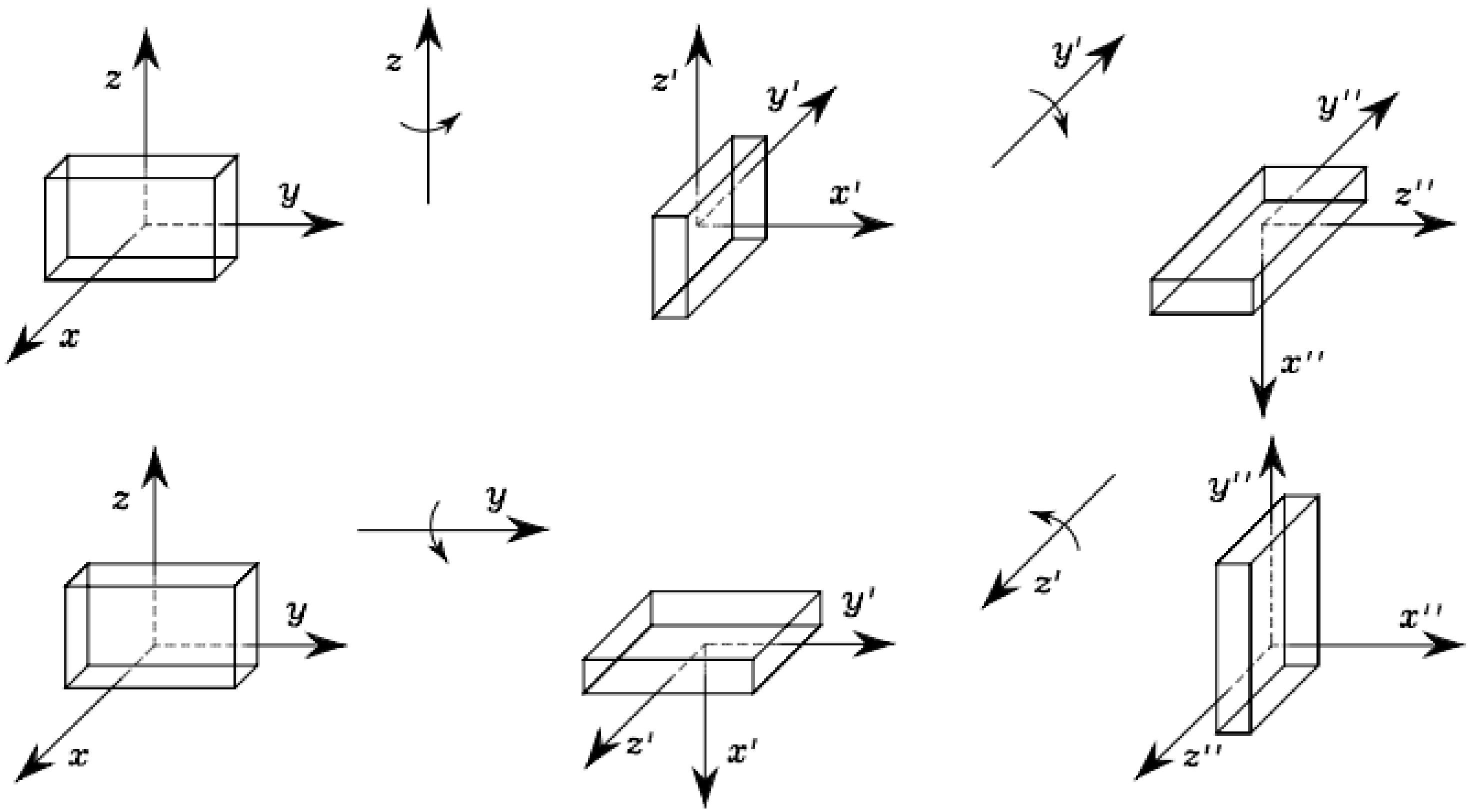
$$\bar{\boldsymbol{R}}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_0^1 \bar{\boldsymbol{R}}_2^1 \boldsymbol{R}_1^0 \quad \rightarrow \quad \bar{\boldsymbol{R}}_2^0 = \bar{\boldsymbol{R}}_2^1 \boldsymbol{R}_1^0$$

- Hence, it can be stated that composition of successive rotations with respect to a fixed frame is obtained by **premultiplication** of the single rotation matrices in the order of the given sequence of rotations.

  - An important issue of composition of rotations is that the matrix product is not commutative.

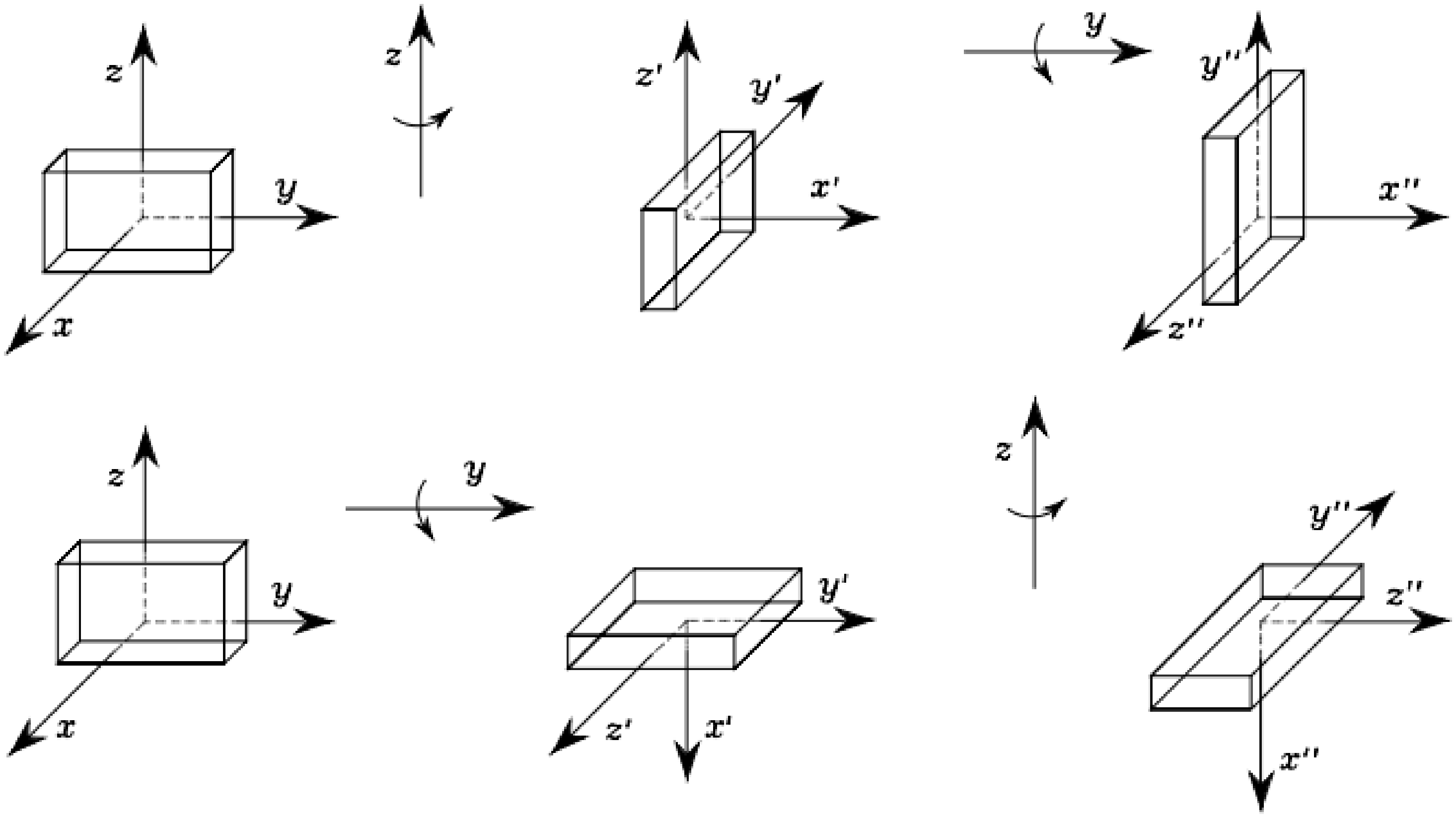
2.3 Composition of Rotation Matrices

- Example 2.3
  - Successive rotations of an object about axes of current frame



2.3 Composition of Rotation Matrices

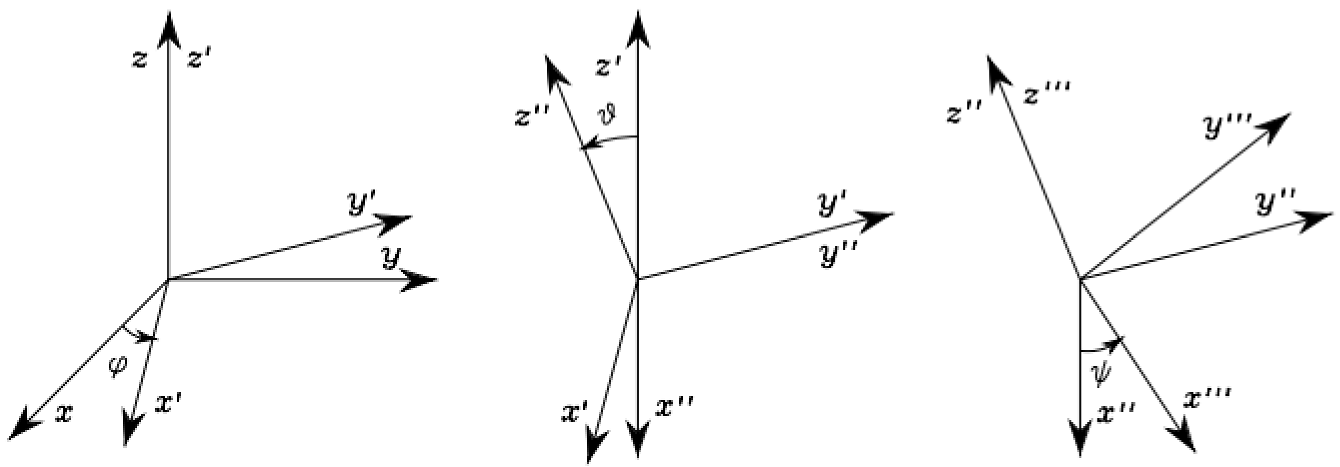
- Example 2.3
  - Successive rotations of an object about axes of fixed frame



2.4 Euler Angles

- A minimal representation of orientation can be obtained by using a set of three angles $[\phi \ \vartheta \ \psi]^T$.

- 2.4.1 ZYZ Angles
  - Rotate the reference frame by the angle ϕ about axis z
  - Rotate the current frame by the angle ϑ about axis y
  - Rotate the current frame by the angle ψ about axis z



2.4 Euler Angles

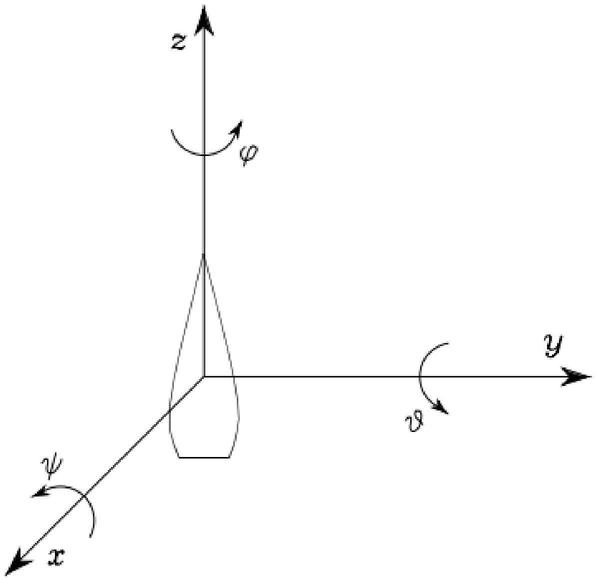
- 2.4.1 ZYZ Angles
  - The resulting frame orientation is obtained by composition of rotations with respect to current frames

$$
\boldsymbol{R}(\phi) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)
$$

$$
= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}
$$

2.4 Euler Angles

- 2.4.2 RPY Angles
  - Representation of orientation in the aeronautical field.
  - These are the ZYX angles, also called Roll–Pitch–Yaw angles, to denote the typical changes of attitude of an aircraft.
  - The angles $[\ \phi\ \ \vartheta\ \ \psi\ ]^T$ represent rotations defined with respect to a fixed frame attached to the center of mass of the aircraft.



2.4 Euler Angles

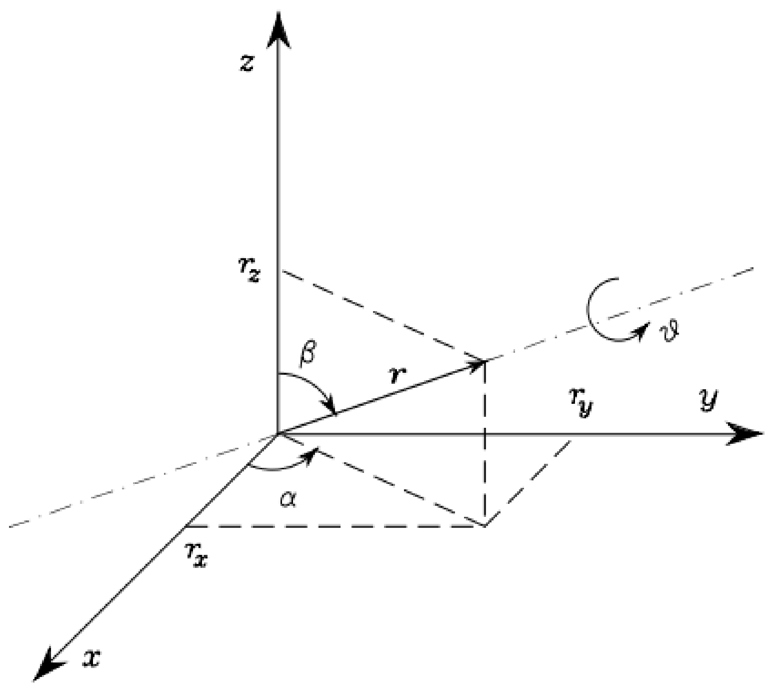
- 2.4.2 RPY Angles
  - Rotate the reference frame by the angle ψ about axis x (yaw)
  - Rotate the reference frame by the angle ϑ about axis y (pitch)
  - Rotate the reference frame by the angle ϕ about axis z (roll)

$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$

2.5 Angle and Axis

- A nonminimal representation: rotation of a given angle about an axis in space (with 4 parameters)
- This can be advantageous in the problem of trajectory planning for a manipulator's end-effector orientation.



2.5 Angle and Axis

- Let $\boldsymbol{r} = [\, r_x \; r_y \; r_z \,]^T$ be the unit vector of a rotation axis with respect to the reference frame O–xyz.
- In order to derive the rotation matrix $R(\vartheta, r)$ expressing the rotation of an angle ϑ about axis r

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha)$$

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z.$$

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \begin{bmatrix} r_x^2(1 - c_\vartheta) + c_\vartheta & r_x r_y(1 - c_\vartheta) - r_z s_\vartheta & r_x r_z(1 - c_\vartheta) + r_y s_\vartheta \\ r_x r_y(1 - c_\vartheta) + r_z s_\vartheta & r_y^2(1 - c_\vartheta) + c_\vartheta & r_y r_z(1 - c_\vartheta) - r_x s_\vartheta \\ r_x r_z(1 - c_\vartheta) - r_y s_\vartheta & r_y r_z(1 - c_\vartheta) + r_x s_\vartheta & r_z^2(1 - c_\vartheta) + c_\vartheta \end{bmatrix}$$

2.5 Angle and Axis

- The inverse problem

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \rightarrow$$

$$\vartheta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\boldsymbol{r} = \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix},$$

  - The three components of $\boldsymbol{r}$ are not independent but are constrained:

$$r_x^2 + r_y^2 + r_z^2 = 1$$

2.6 Unit Quaternion

- The drawbacks of the angle/axis representation can be overcome by a different four-parameter representation; namely, the **Unit Quaternion**

$$\mathcal{Q} = \{\eta, \boldsymbol{\epsilon}\}$$

$$\eta = \cos\frac{\vartheta}{2}$$

→

$$\boldsymbol{\epsilon} = \sin\frac{\vartheta}{2}\boldsymbol{r}$$

2.6 Unit Quaternion

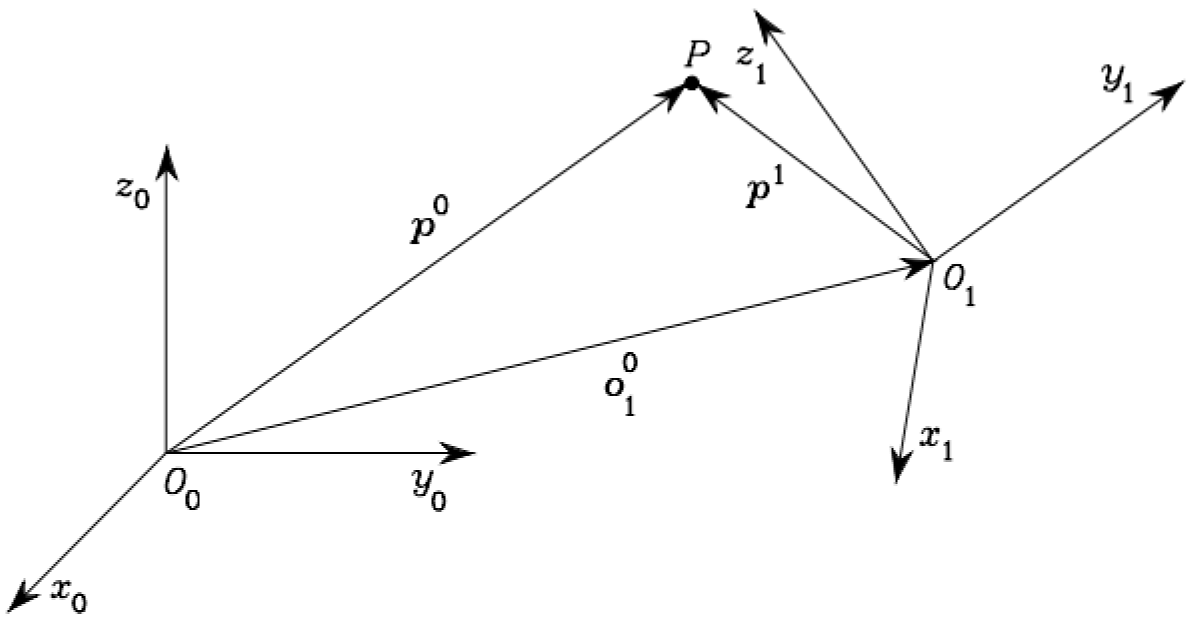
- η : the scalar part of the quaternion while
- $\varepsilon = [\varepsilon_x \; \varepsilon_y \; \varepsilon_z]^T$: the vector part of the quaternion.

$$\eta^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 = 1$$

$$\rightarrow \quad \boldsymbol{R}(\eta, \boldsymbol{\epsilon}) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}$$

2.7 Homogeneous Transformations

- The position of a rigid body in space:
  - Position of a point on the body with respect to a reference frame (translation)
  - Components of the unit vectors (orientation) of a frame attached to the body with respect to the same reference frame (rotation)



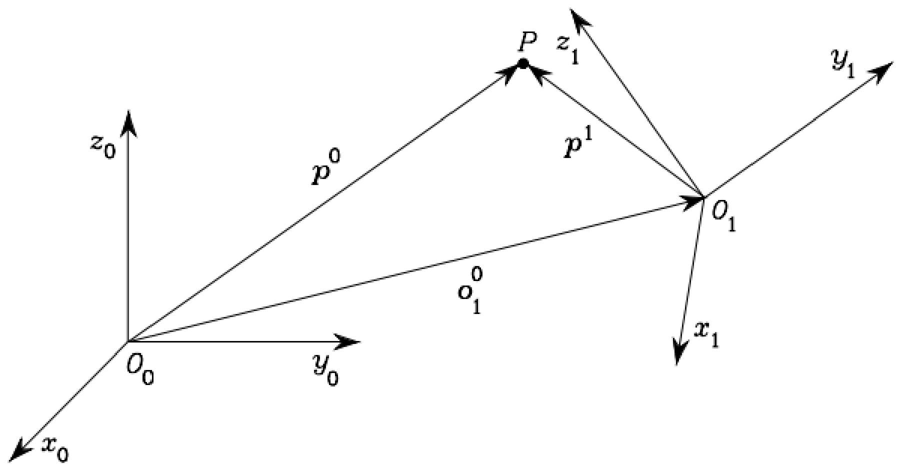
2.7 Homogeneous Transformations

- Coordinate transformation (translation + rotation) of a bound vector between two frames:
  - ✓ R_0^1 : Rotation matrix of Frame 1 with respect to Frame 0

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

- The inverse transformation:

$$\boldsymbol{p}^1 = -\boldsymbol{R}_1^{0\,T} \boldsymbol{o}_1^0 + \boldsymbol{R}_1^{0\,T} \boldsymbol{p}^0 \quad \longrightarrow \quad \boldsymbol{p}^1 = -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 + \boldsymbol{R}_0^1 \boldsymbol{p}^0$$



2.7 Homogeneous Transformations

- The **Homogeneous Representation** of a generic vector **p** : (**p**˜)
  - In order to achieve a compact representation of the relationship between the coordinates of the same point in two different frames

$$\widetilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix} \rightarrow \begin{aligned} \widetilde{\boldsymbol{p}}^0 &= \boldsymbol{A}_1^0 \widetilde{\boldsymbol{p}}^1 \\ \widetilde{\boldsymbol{p}}^1 &= \boldsymbol{A}_0^1 \widetilde{\boldsymbol{p}}^0 = \left(\boldsymbol{A}_1^0\right)^{-1} \widetilde{\boldsymbol{p}}^0 \end{aligned}$$

  - *Homogeneous Transformation Matrix*

$$\boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

2.7 Homogeneous Transformations

- *Homogeneous Transformation Matrix*

$$A_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0^T & 1 \end{bmatrix}$$

$$A_0^1 = \begin{bmatrix} R_1^{0T} & -R_1^{0T} o_1^0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_0^1 & -R_0^1 o_1^0 \\ 0^T & 1 \end{bmatrix}$$

- Notice that:

$$A^{-1} \neq A^T$$

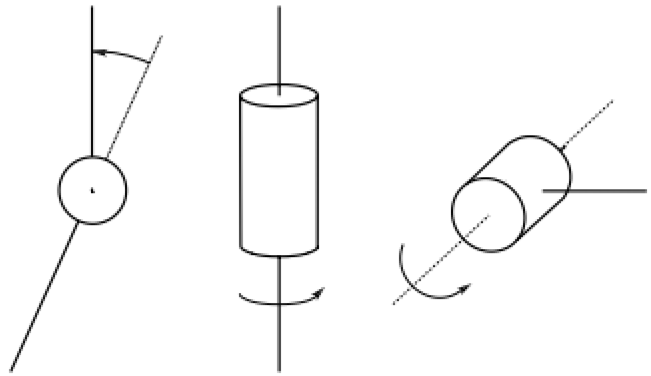
$$\widetilde{p}^0 = A_1^0 A_2^1 \ldots A_n^{n-1} \widetilde{p}^n$$

2.8 Direct Kinematics

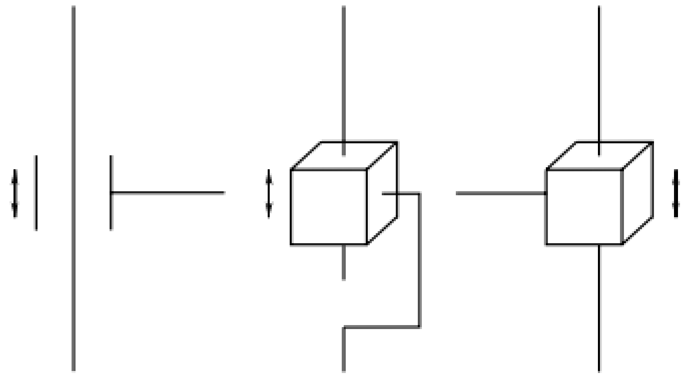
- A manipulator:
  - Series of rigid bodies (links) connected by means of kinematic pairs or joints

- Joints:
  - Revolute
  - Prismatic

REVOLUTE



PRISMATIC



2.8 Direct Kinematics

- The whole structure forms a Kinematic Chain.
  - ✓ One end of the chain is constrained to a base.
  - ✓ The other end is an end-effector (gripper, tool)

- Open kinematic chain (only one sequence of links connecting the two ends)
- Closed kinematic chain (a sequence of links forms a loop)

- Characterized by a number of degrees of freedom (DOFs)
  - ✓ Uniquely determine its posture.
  - ✓ Each DOF is typically associated with a joint articulation and constitutes a joint variable

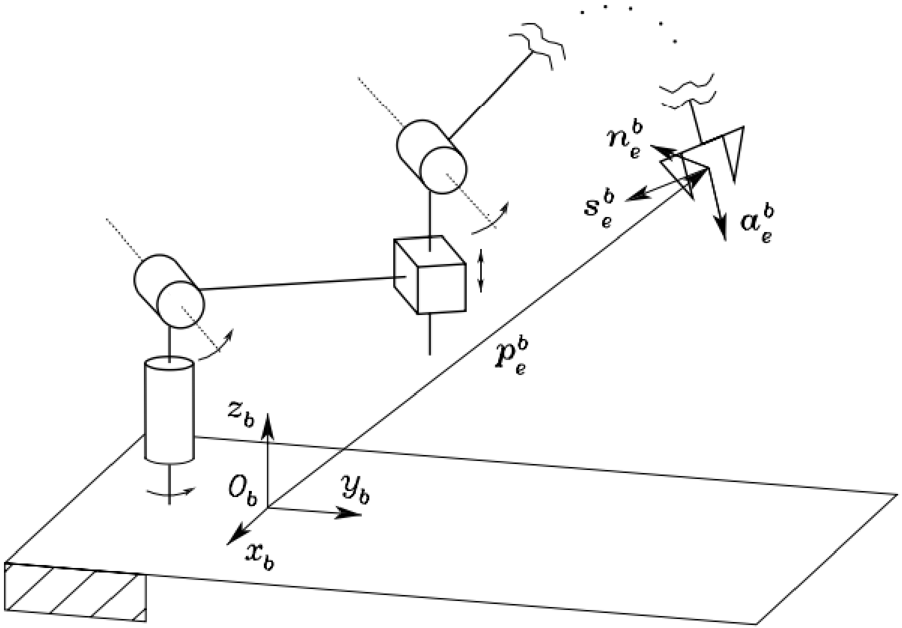
- **Direct kinematics:**
  - Compute the pose of the end-effector as a function of the joint variables

2.8 Direct Kinematics

- Direct kinematics function homogeneous transformation matrix

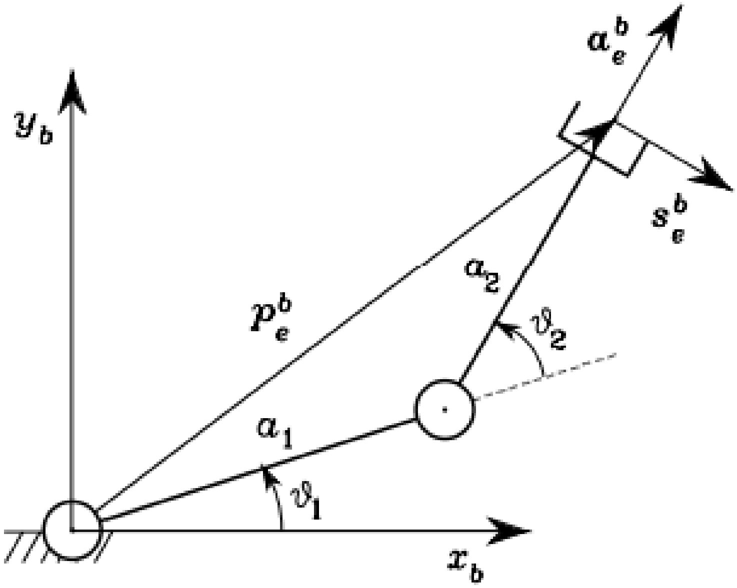
$$T_e^b(q) = \begin{bmatrix} n_e^b(q) & s_e^b(q) & a_e^b(q) & p_e^b(q) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - n_e, s_e, a_e and p_e are a function of q



2.8 Direct Kinematics

- Example 2.4
  - Two-link planar arm

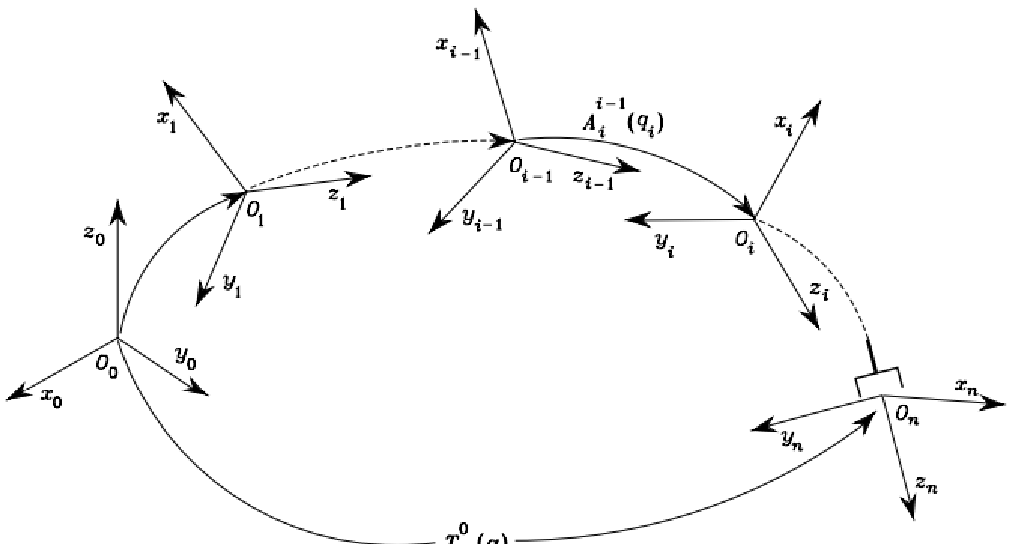


$$T_e^b(q) = \begin{bmatrix} n_e^b & s_e^b & a_e^b & p_e^b \\ \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & s_{12} & c_{12} & a_1c_1 + a_2c_{12} \\ 0 & -c_{12} & s_{12} & a_1s_1 + a_2s_{12} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.8 Direct Kinematics

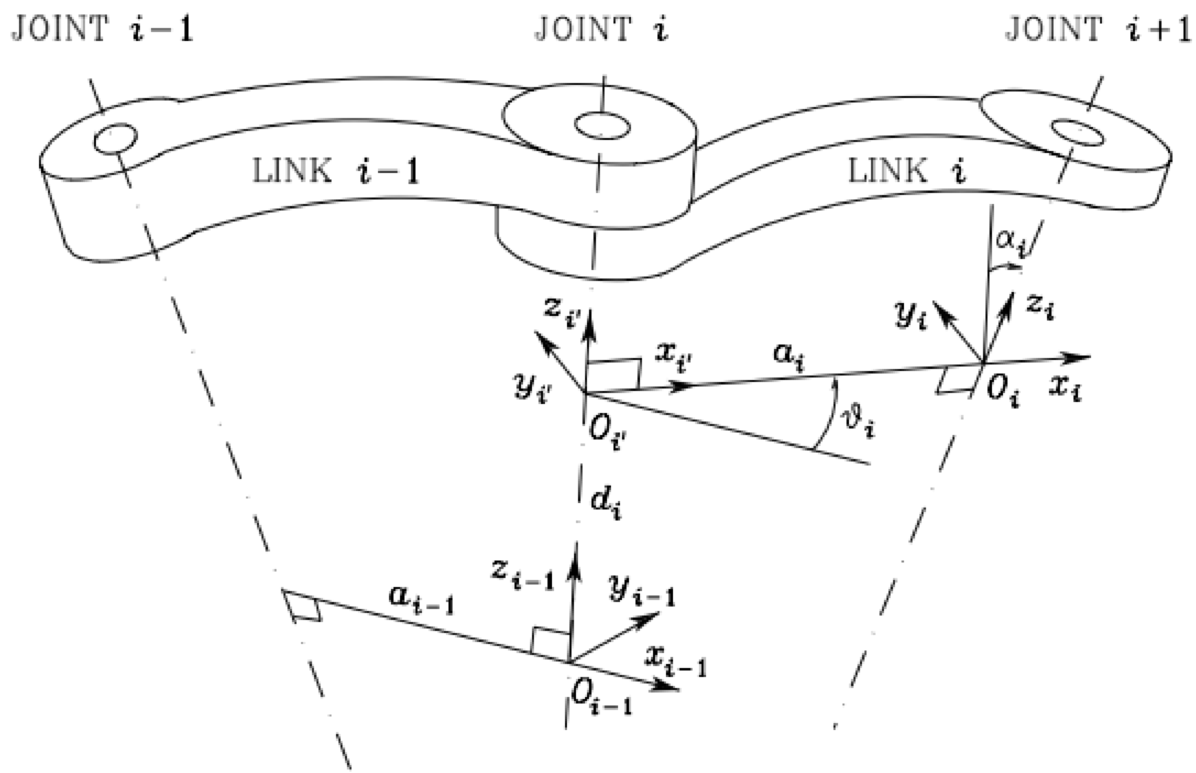
- 2.8.1 Open Chain
  - An open-chain manipulator constituted by n + 1 links connected by n joints
  - Define a coordinate frame attached to each link, from Link 0 to Link n
  - The coordinate transformation describing the position and orientation of Frame n with respect to Frame 0:

$$\boldsymbol{T}_n^0(\boldsymbol{q}) = \boldsymbol{A}_1^0(q_1)\boldsymbol{A}_2^1(q_2)\ldots\boldsymbol{A}_n^{n-1}(q_n) \rightarrow \boldsymbol{T}_e^b(\boldsymbol{q}) = \boldsymbol{T}_0^b\boldsymbol{T}_n^0(\boldsymbol{q})\boldsymbol{T}_e^n$$



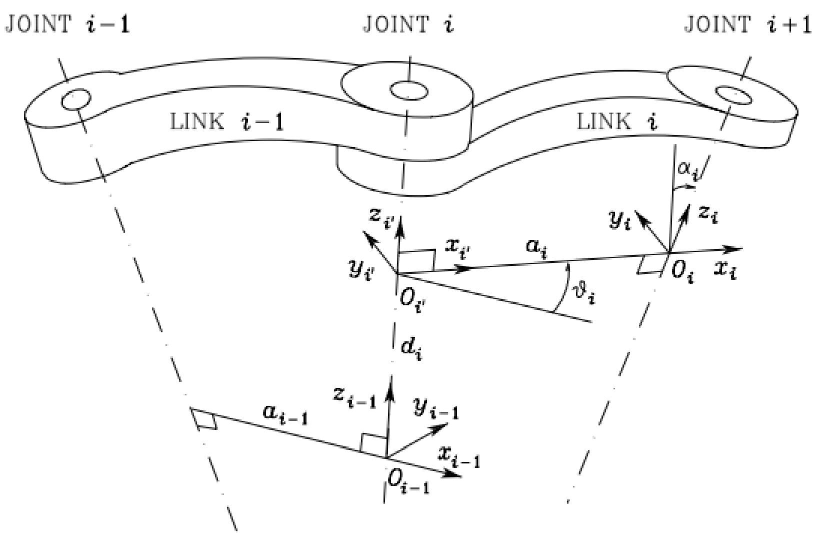
2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - A systematic, general method is to be derived to define the relative position and orientation of two consecutive links



2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Let Axis i denote the axis of the joint connecting Link i − 1 to Link i
  - The Denavit–Hartenberg convention (DH) is adopted to define link Frame i:
    - ✓ Choose axis zi along the axis of Joint i + 1
    - ✓ Locate the origin O_i and O_i’
    - ✓ Choose axis x_i along the common normal to axes z_{i-1} and z_i (from Joint i to Joint i + 1)
    - ✓ Choose axis y_i so as to complete a right-handed frame.



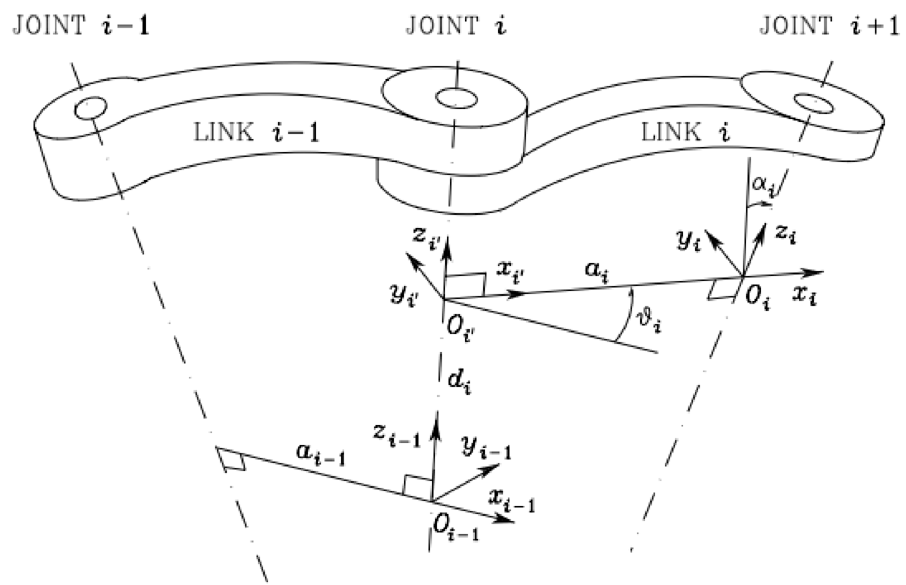
2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - The Denavit–Hartenberg convention gives a nonunique definition of the link frame in the following cases:
    - For Frame 0, only the direction of axis z_0 is specified; O_0 and x_0 can be arbitrarily chosen
    - For Frame n (no Joint n+1) z_n is not uniquely defined while x_n has to be normal to axis z_{n-1}

      (Typically, Joint n is revolute, and thus z_n is to be aligned with the direction of z_{n-1})
    - When two consecutive axes are parallel, the common normal is not uniquely defined
    - When two consecutive axes intersect, the direction of xi is arbitrary
    - When Joint i is prismatic, the direction of z_{i-1} is arbitrary

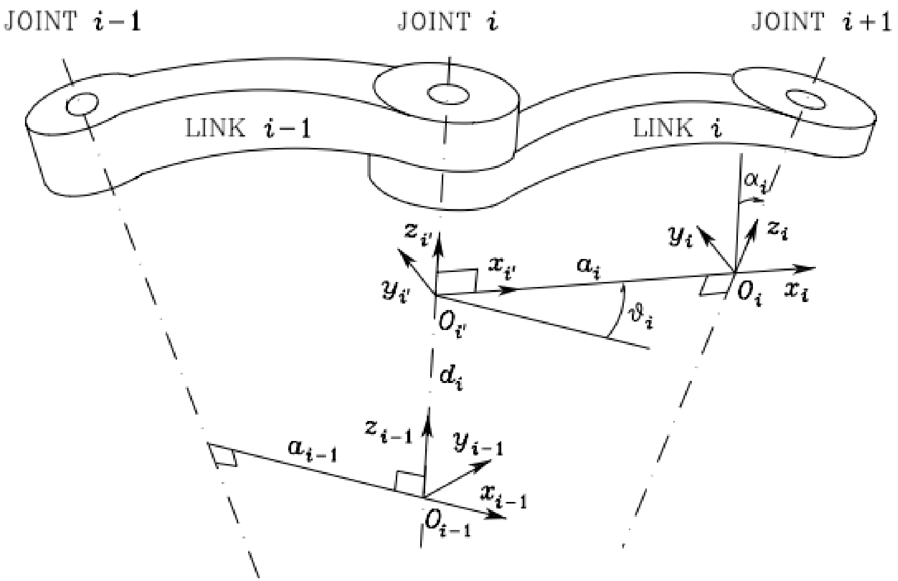
2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Parameters:
    - ✓ $\boldsymbol{a_i}$: Distance between O_i and O_i’
    - ✓ $\boldsymbol{d_i}$: Coordinate of O_i along z_{i-1}
    - ✓ $\boldsymbol{\alpha_i}$: Angle between axes z_{i-1} and z_i about axis xi (positive: counter-clockwise)
    - ✓ $\boldsymbol{\vartheta_i}$: Angle between axes x_{i-1} and x_i about axis z_{i-1} (positive: counter-clockwise)



2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Two of the four parameters (a_i and α_i) are always constant and depend only on the geometry of connection between consecutive joints.
  - ***If Joint i is revolute the variable is ϑ_i***
  - ***If Joint i is prismatic the variable is d_i***



2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Coordinate transformation between Frame i and Frame i − 1:
    1. Choose a frame aligned with Frame i − 1
    2. Translate the chosen frame by d_i along axis z_{i-1} and rotate it by ϑ_i about axis z_{i-1}

$$\boldsymbol{A}_{i'}^{i-1} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} & 0 & 0 \\ s_{\vartheta_i} & c_{\vartheta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Coordinate transformation between Frame i and Frame i − 1:

3. Translate the frame aligned with Frame i' by a_i along x_i and rotate it by α_i about x_i

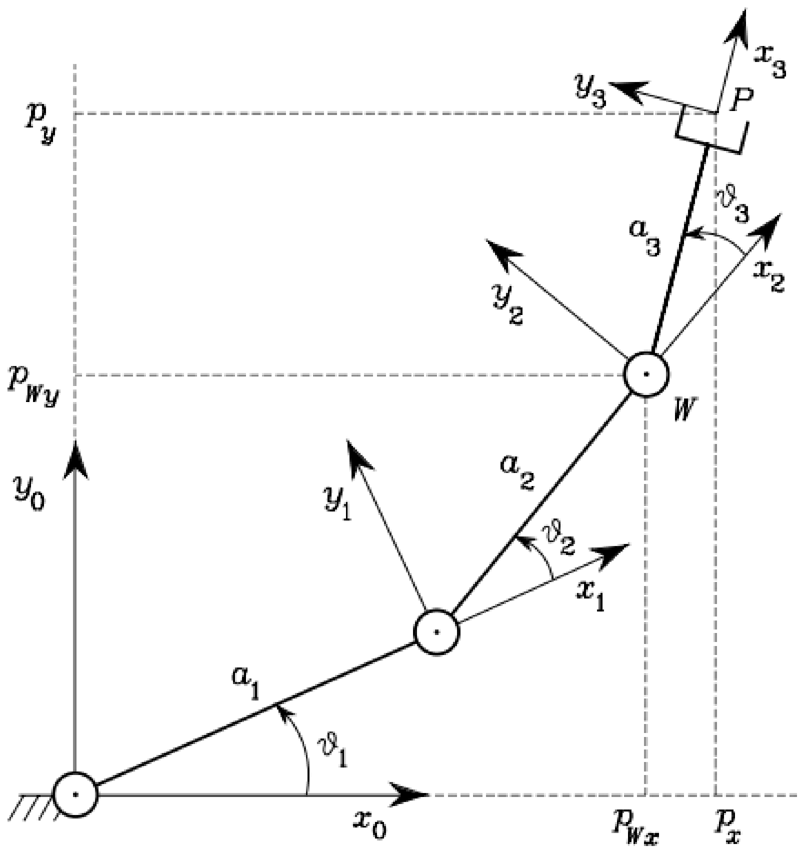
$$\boldsymbol{A}_i^{i'} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Post-multiplicate the single transformations:

$$\boldsymbol{A}_i^{i-1}(q_i) = \boldsymbol{A}_{i'}^{i-1}\boldsymbol{A}_i^{i'} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i}c_{\alpha_i} & s_{\vartheta_i}s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i}c_{\alpha_i} & -c_{\vartheta_i}s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.9 Kinematics of Typical Manipulator Structures

- 2.9.1 Three-link Planar Arm

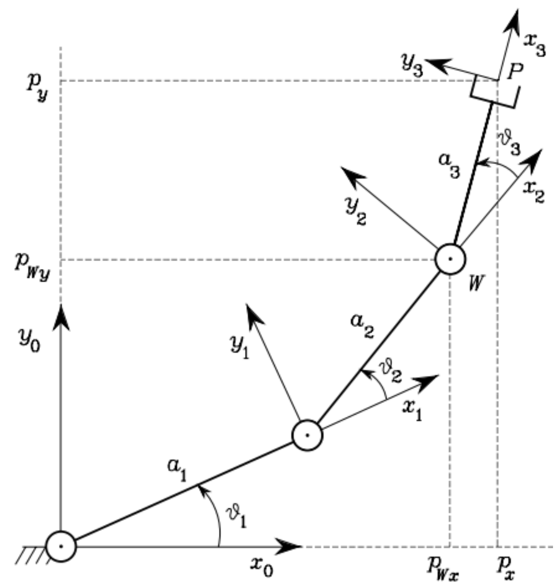


2.9 Kinematics of Typical Manipulator Structures

- 2.9.1 Three-link Planar Arm
  - DH Parameters

Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, 2, 3$$



2.9 Kinematics of Typical Manipulator Structures

- 2.9.1 Three-link Planar Arm
  - All joints are revolute:

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0\boldsymbol{A}_2^1\boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

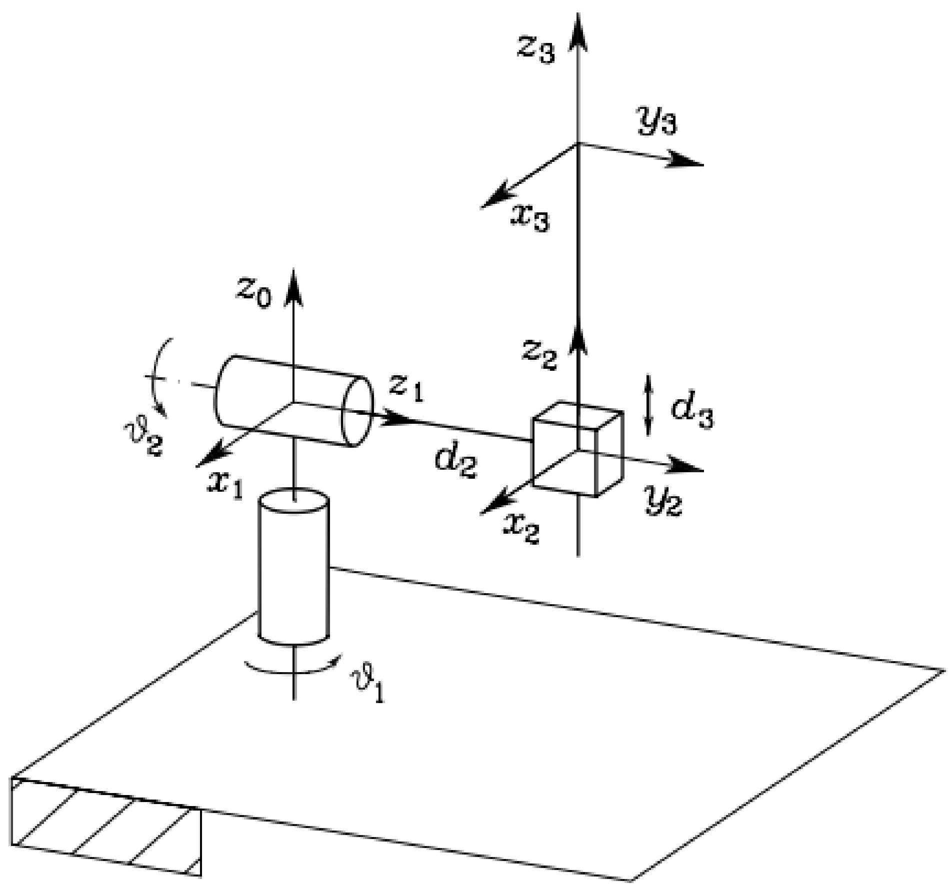
$$\boldsymbol{q} = [\vartheta_1 \quad \vartheta_2 \quad \vartheta_3]^T$$

  - End-effector frame:

$$\boldsymbol{T}_e^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.9 Kinematics of Typical Manipulator Structures

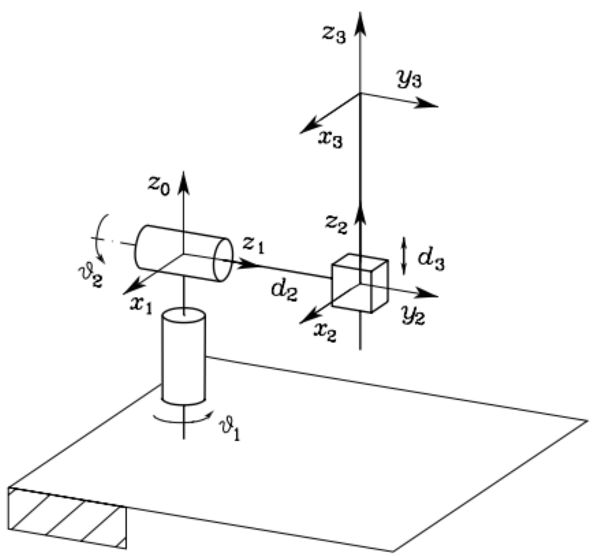
- 2.9.3 Spherical Arm



2.9 Kinematics of Typical Manipulator Structures

- 2.9.3 Spherical Arm
  - DH Parameters

Link	a_i	α_i	d_i	ϑ_i
1	0	$-\pi/2$	0	ϑ_1
2	0	$\pi/2$	d_2	ϑ_2
3	0	0	d_3	0



2.9 Kinematics of Typical Manipulator Structures

- 2.9.3 Spherical Arm
  - The homogeneous transformation matrices:

$$\boldsymbol{A}_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_2^1(\vartheta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

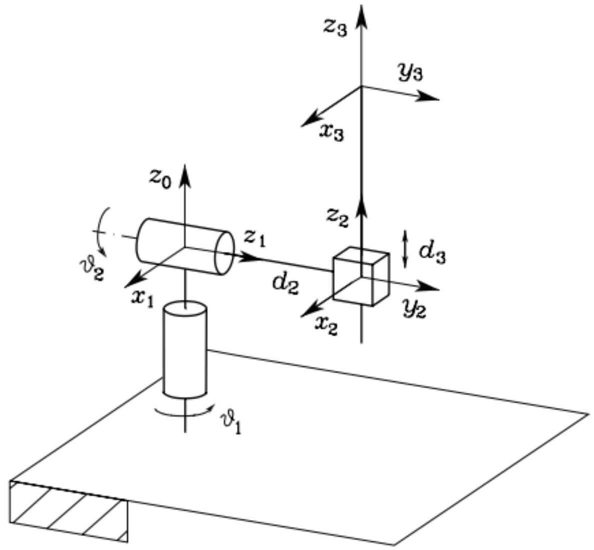
$$\boldsymbol{A}_3^2(d_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2.9 Kinematics of Typical Manipulator Structures

- 2.9.3 Spherical Arm
  - The direct kinematics function:

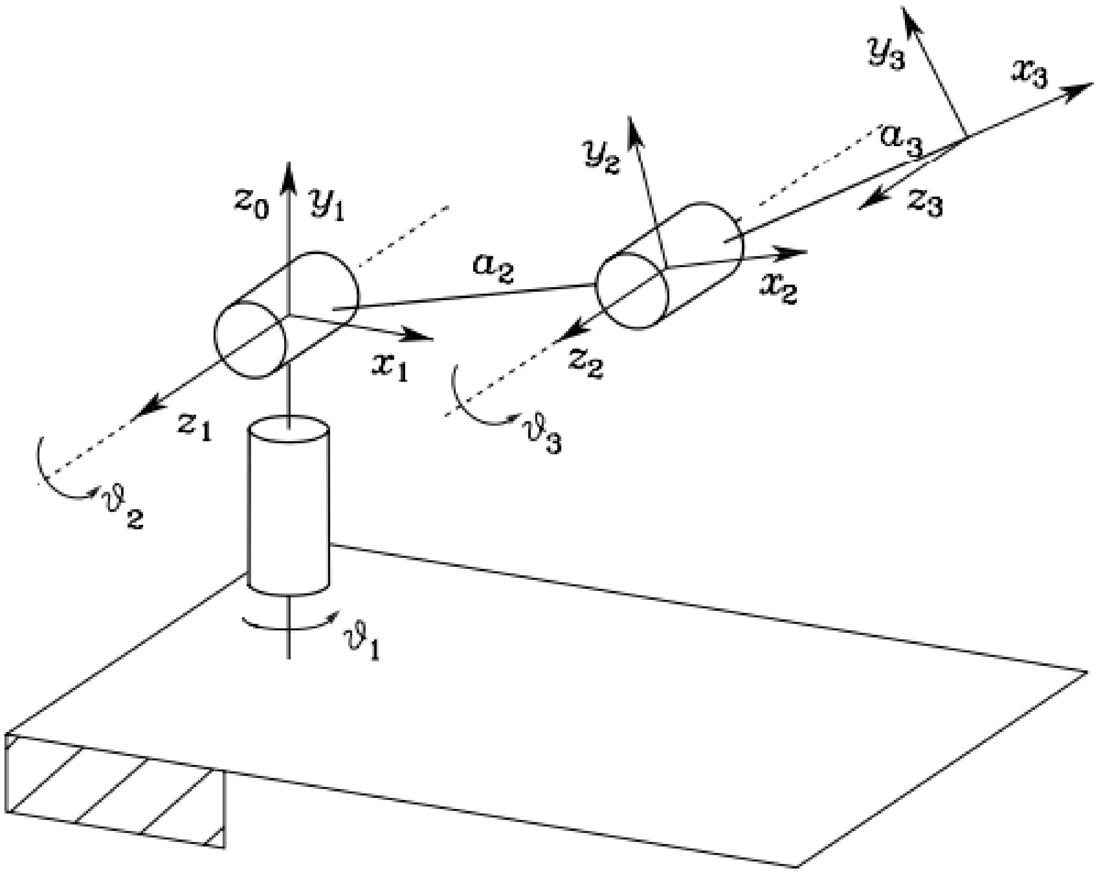
$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 s_2 d_3 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 s_2 d_3 + c_1 d_2 \\ -s_2 & 0 & c_2 & c_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{q} = [\vartheta_1 \quad \vartheta_2 \quad d_3]^T$$



2.9 Kinematics of Typical Manipulator Structures

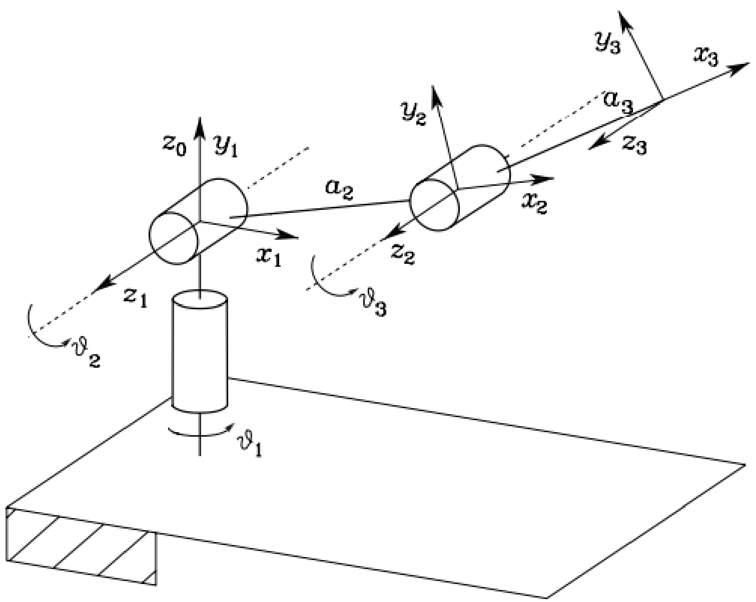
- 2.9.4 Anthropomorphic Arm



2.9 Kinematics of Typical Manipulator Structures

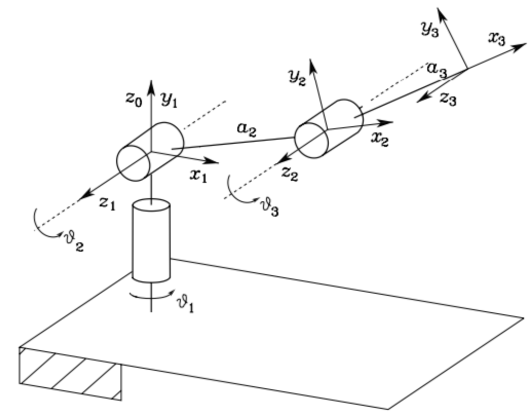
- 2.9.4 Anthropomorphic Arm
  - DH Parameters:

Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3



2.9 Kinematics of Typical Manipulator Structures

- 2.9.4 Anthropomorphic Arm



$$\boldsymbol{A}_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

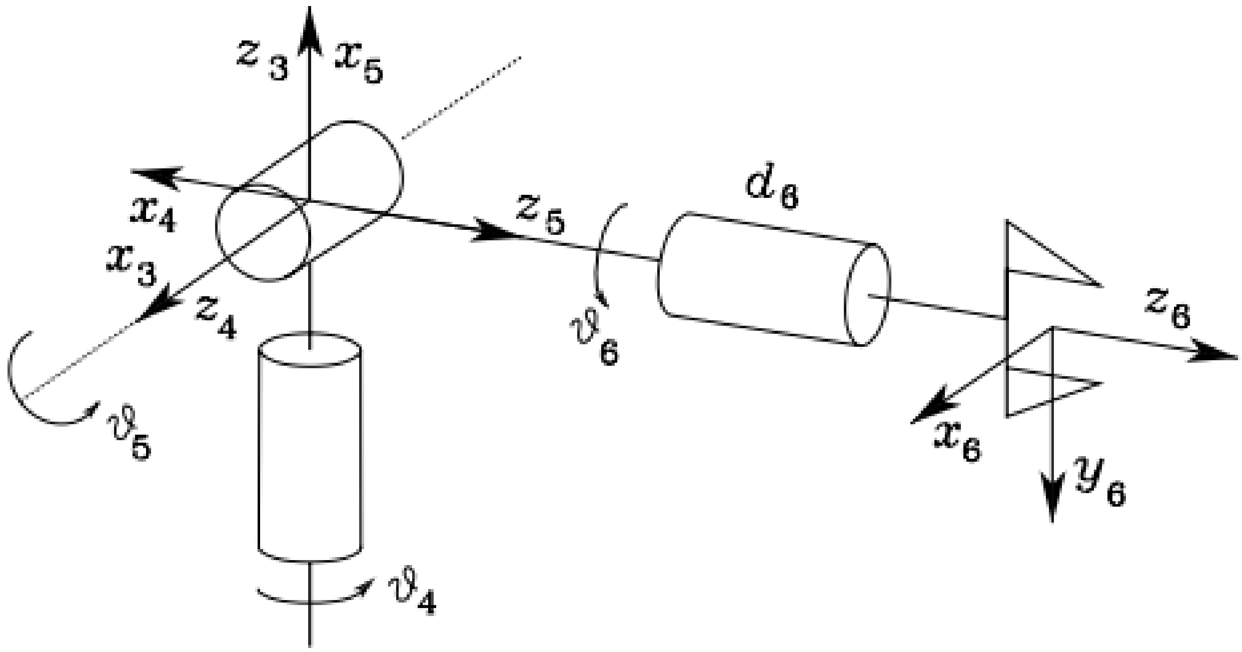
$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 2, 3$$

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{q} = [\,\vartheta_1 \quad \vartheta_2 \quad \vartheta_3\,]^T$$

2.9 Kinematics of Typical Manipulator Structures

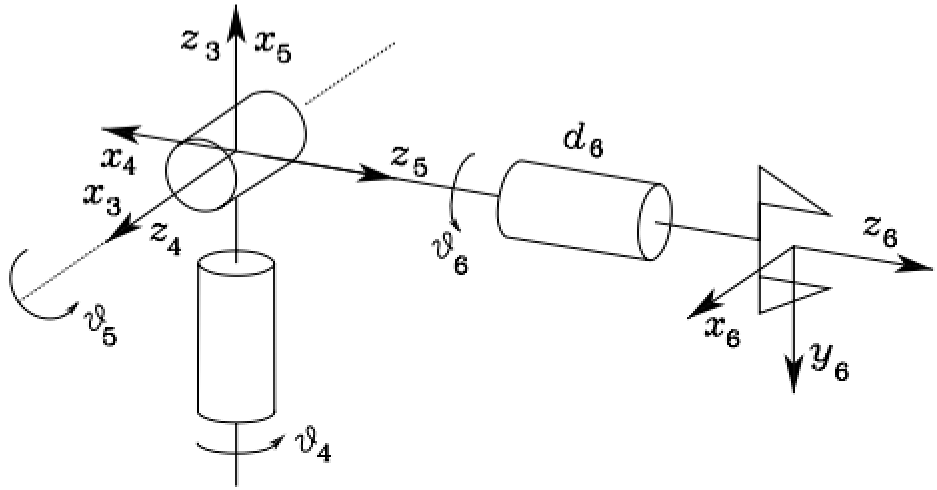
- 2.9.5 Spherical Wrist



2.9 Kinematics of Typical Manipulator Structures

- 2.9.5 Spherical Wrist
  - DH Parameters:

Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

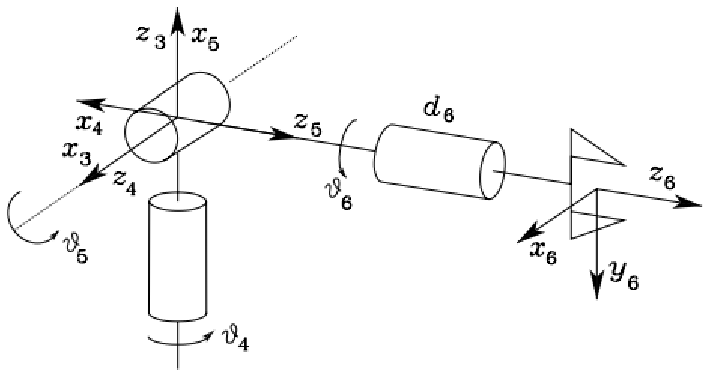


2.9 Kinematics of Typical Manipulator Structures

- 2.9.5 Spherical Wrist

$$\boldsymbol{A}_4^3(\vartheta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_5^4(\vartheta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_6^5(\vartheta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

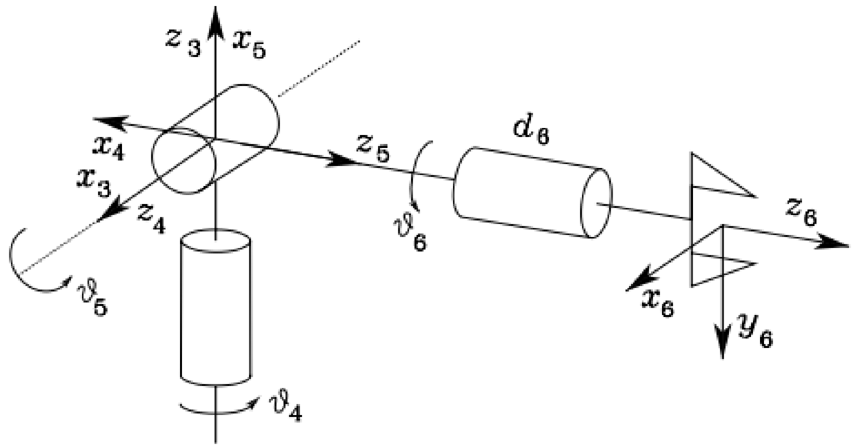


2.9 Kinematics of Typical Manipulator Structures

- 2.9.5 Spherical Wrist

$$\boldsymbol{T}_6^3(\boldsymbol{q}) = \boldsymbol{A}_4^3\boldsymbol{A}_5^4\boldsymbol{A}_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

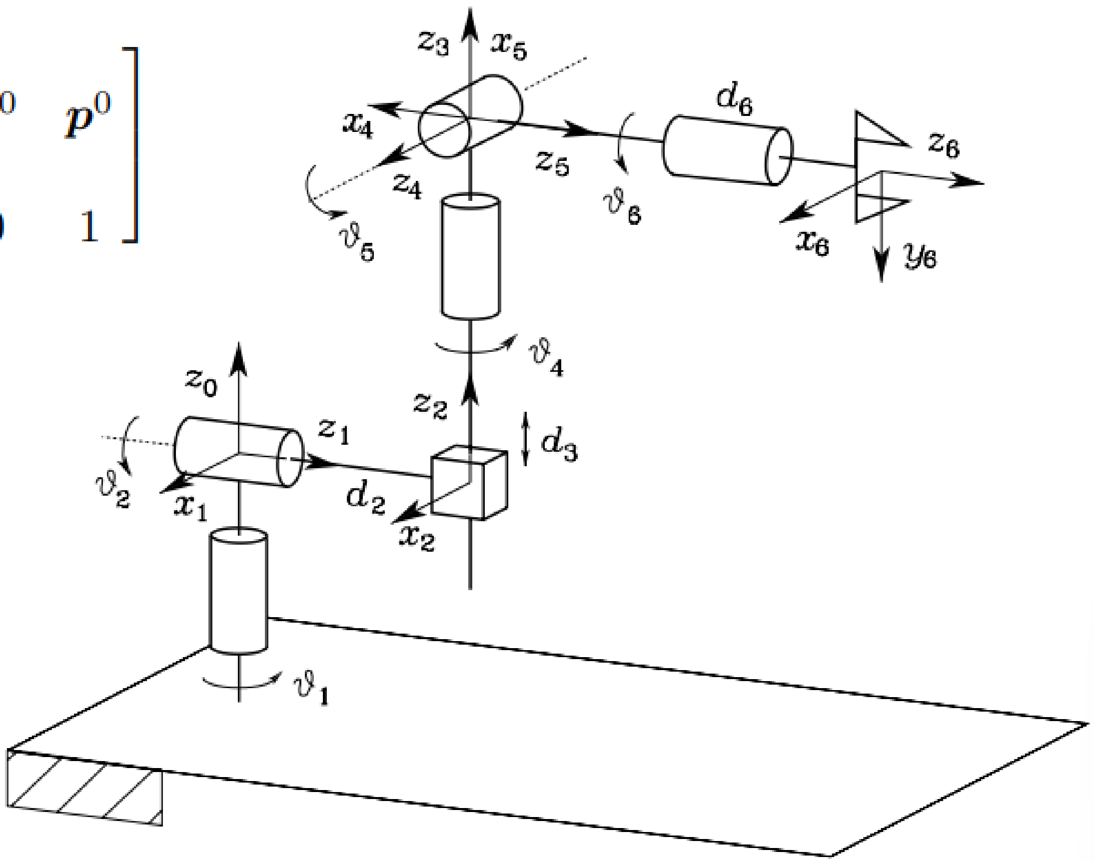
$$\boldsymbol{q} = [\,\vartheta_4 \quad \vartheta_5 \quad \vartheta_6\,]^T$$



2.9 Kinematics of Typical Manipulator Structures

- 2.9.6 Stanford Manipulator

$$\boldsymbol{T}_6^0 = \boldsymbol{T}_3^0 \boldsymbol{T}_6^3 = \begin{bmatrix} \boldsymbol{n}^0 & \boldsymbol{s}^0 & \boldsymbol{a}^0 & \boldsymbol{p}^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



2.9 Kinematics of Typical Manipulator Structures

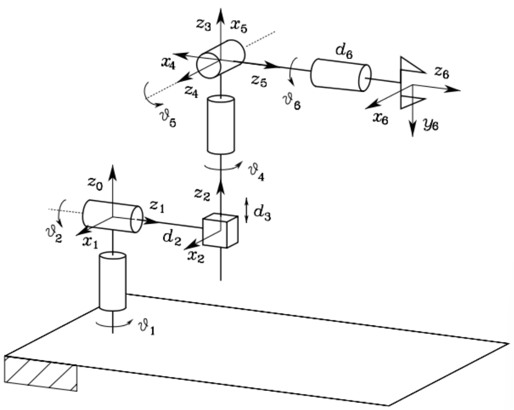
- 2.9.6 Stanford Manipulator

$$\boldsymbol{p}_6^0 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + \big(c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5\big) d_6 \\ s_1 s_2 d_3 + c_1 d_2 + \big(s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5\big) d_6 \\ c_2 d_3 + (-s_2 c_4 s_5 + c_2 c_5) d_6 \end{bmatrix}$$

$$\boldsymbol{n}_6^0 = \begin{bmatrix} c_1\big(c_2(c_4 c_5 c_6 - s_4 s_6) - s_2 s_5 c_6\big) - s_1(s_4 c_5 c_6 + c_4 s_6) \\ s_1\big(c_2(c_4 c_5 c_6 - s_4 s_6) - s_2 s_5 c_6\big) + c_1(s_4 c_5 c_6 + c_4 s_6) \\ -s_2(c_4 c_5 c_6 - s_4 s_6) - c_2 s_5 c_6 \end{bmatrix}$$

$$\boldsymbol{s}_6^0 = \begin{bmatrix} c_1\big(-c_2(c_4 c_5 s_6 + s_4 c_6) + s_2 s_5 s_6\big) - s_1(-s_4 c_5 s_6 + c_4 c_6) \\ s_1\big(-c_2(c_4 c_5 s_6 + s_4 c_6) + s_2 s_5 s_6\big) + c_1(-s_4 c_5 s_6 + c_4 c_6) \\ s_2(c_4 c_5 s_6 + s_4 c_6) + c_2 s_5 s_6 \end{bmatrix}$$

$$\boldsymbol{a}_6^0 = \begin{bmatrix} c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5 \\ s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$



2.10 Joint Space and Operational Space

- Direct Kinematics:
  - Position and orientation of the end-effector frame to be expressed as a function of the joint variables with respect to the base frame.
  - This is quite easy for the position, but quite difficult for orientation (9 components must be guaranteed to satisfy the orthonormality constraints)

- The end-effector pose can be given by a minimal number of coordinates and minimal representation (Euler angles) describing the rotation

$$\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{p}_e \\ \boldsymbol{\phi}_e \end{bmatrix}$$

  - p_e : End-effector position
  - φ_e : End-effector orientation

2.10 JOINT SPACE AND OPERATIONAL SPACE

- The vector X_e is defined in the space in which the manipulator task is specified; hence, this space is typically called ***operational space***.

$$x_e = \begin{bmatrix} p_e \\ \phi_e \end{bmatrix}$$

- On the other hand, the ***joint space*** (configuration space) denotes the space in which the (n×1) vector of joint variables

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

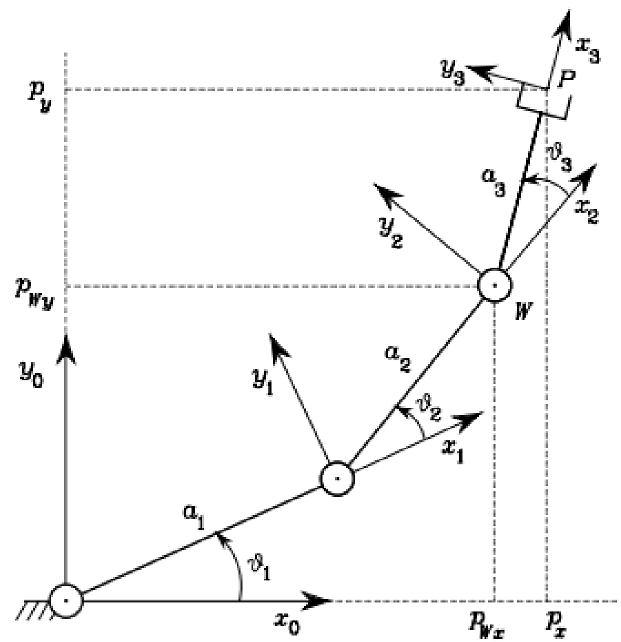
  - For a revolute joint: $q_i = \vartheta_i$
  - For a prismatic joint: $q_i = d_i$

- ***Direct Kinematics Equation:*** $x_e = k(q)$

2.10 Joint Space and Operational Space

- Example 2.5

$$\boldsymbol{x}_e = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \boldsymbol{k}(\boldsymbol{q}) = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

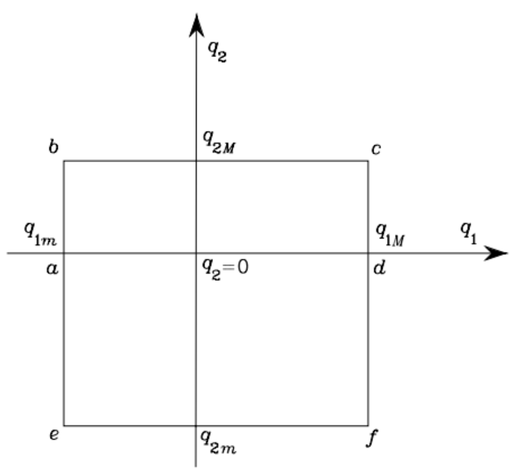


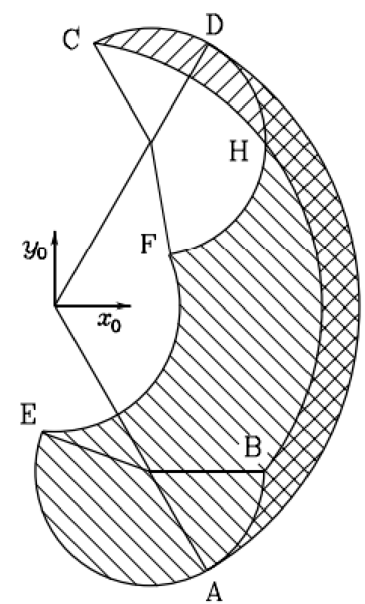
- 3 joint space variables allow specification of at most 3 independent operational space variables.
- If orientation is of no concern, there is *kinematic redundancy.*

2.10 Joint Space and Operational Space

- 2.10.1 Workspace
  - The region described by the origin of the end-effector frame when all the manipulator joints execute all possible motions
  - This volume is finite, closed, connected and is defined by its bordering surface

- Example 2.6
  - The simple two-link planar arm





2.10 Joint Space and Operational Space

- 2.10.2 Kinematic Redundancy
- Kinematically Redundant:
  - When number of DOFs is greater than the number of variables that are necessary to describe a given task

- A manipulator is intrinsically redundant when the dimension of the operational space is smaller than the dimension of the joint space ($m < n$)

- Redundancy is a concept relative to the task assigned to the manipulator.

2.12 Inverse Kinematics Problem

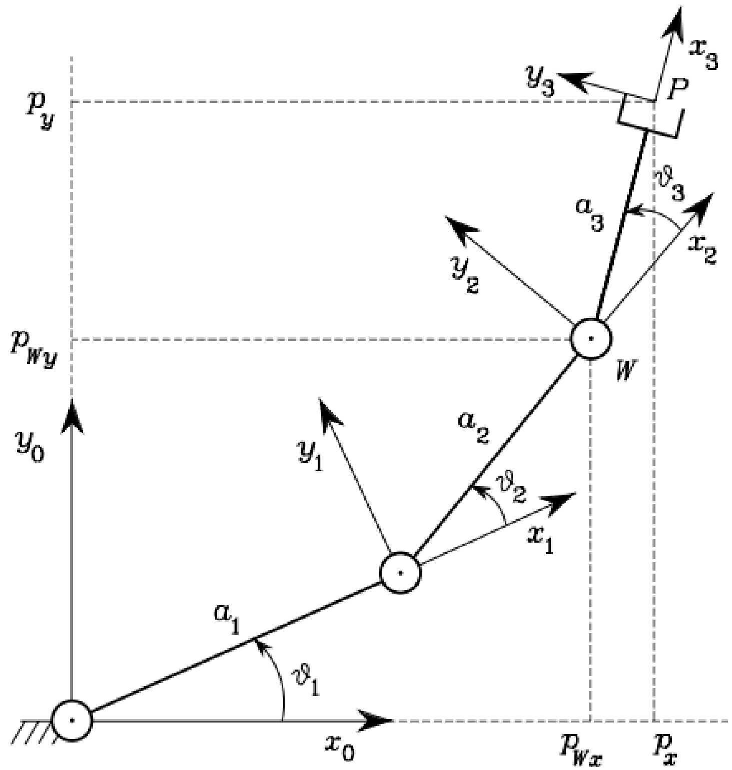
- The inverse kinematics problem consists of the determination of the joint variables corresponding to a given end-effector position and orientation.

- It transforms the motion specifications, assigned to the end-effector in the operational space, into the corresponding joint space motions that allow execution of the desired motion.

- The inverse kinematics problem is much more complex:
  - The equations to solve are in general nonlinear
  - Multiple solutions may exist
  - Infinite solutions may exist
  - There might be no admissible solutions

2.12 Inverse Kinematics Problem

- 2.12.1 Solution of Three-link Planar Arm
  - The end-effector position and orientation in terms of a minimal number of parameters:
    - ✓ The two coordinates p_x, p_y
    - ✓ The angle φ with axis x_0



2.12 Inverse Kinematics Problem

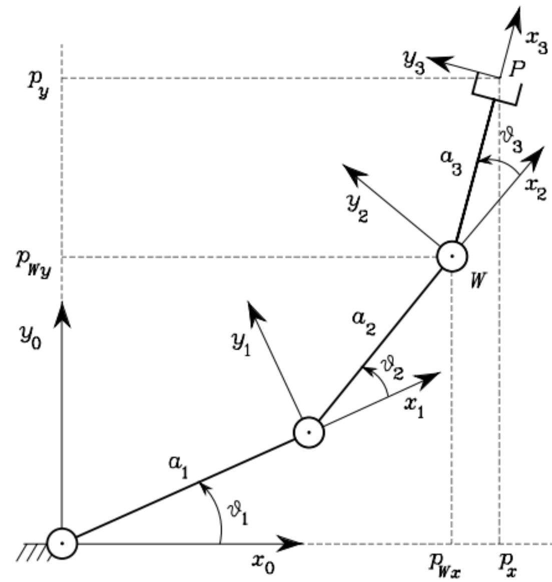
- 2.12.1 Solution of Three-link Planar Arm
  - *Algebraic solution technique*

$$\phi = \vartheta_1 + \vartheta_2 + \vartheta_3$$

$$p_{Wx} = p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12}$$
$$p_{Wy} = p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12}$$

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 + 2a_1 a_2 c_2$$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}$$



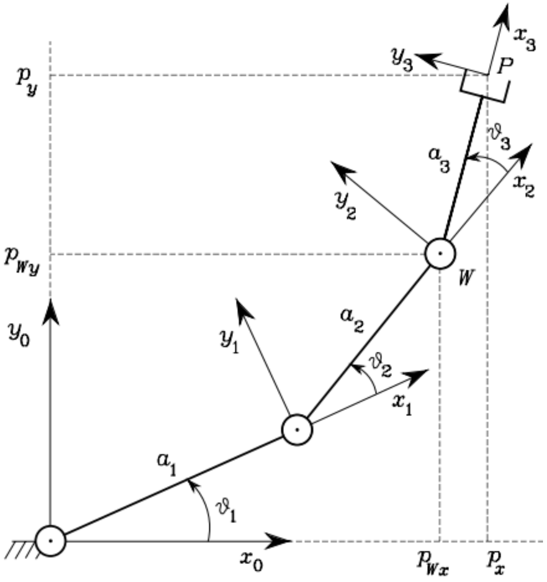
2.12 Inverse Kinematics Problem

- 2.12.1 Solution of Three-link Planar Arm
  - *Algebraic solution technique*

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

$$\rightarrow \quad s_2 = \pm\sqrt{1 - c_2^2}$$

$$\rightarrow \quad \vartheta_2 = \text{Atan2}(s_2, c_2)$$



2.12 Inverse Kinematics Problem

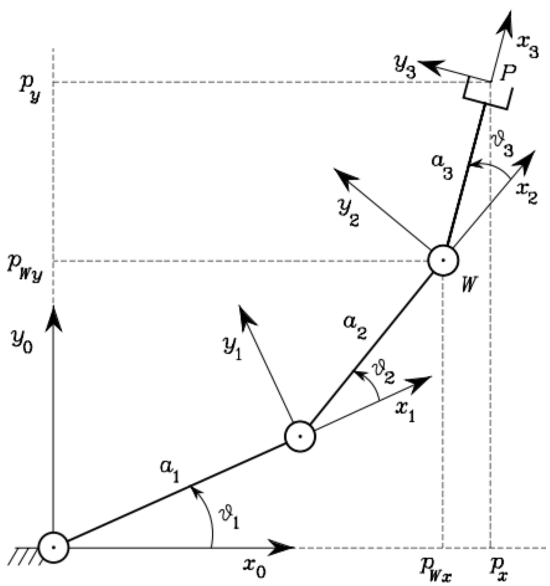
- 2.12.1 Solution of Three-link Planar Arm
  - *Algebraic solution technique*

$$s_1 = \frac{(a_1 + a_2 c_2) p_{Wy} - a_2 s_2 p_{Wx}}{p_{Wx}^2 + p_{Wy}^2}$$

$$c_1 = \frac{(a_1 + a_2 c_2) p_{Wx} + a_2 s_2 p_{Wy}}{p_{Wx}^2 + p_{Wy}^2}$$

$$\vartheta_1 = \text{Atan2}(s_1, c_1)$$

$$\vartheta_3 = \phi - \vartheta_1 - \vartheta_2$$



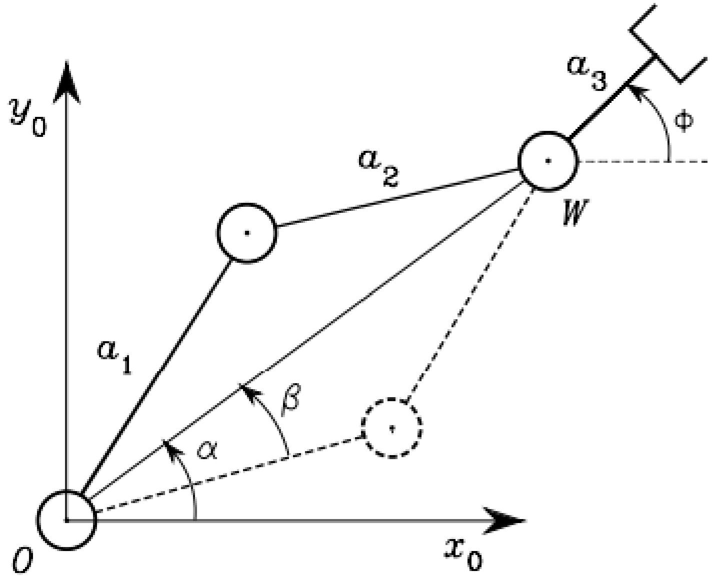
2.12 Inverse Kinematics Problem

- 2.12.1 Solution of Three-link Planar Arm
  - *Geometric solution technique*
    - ✓ The application of the cosine theorem to the triangle formed by links a_1, a_2 and the segment connecting points W and O

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 - 2a_1 a_2 \cos(\pi - \vartheta_2)$$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

$$\vartheta_2 = \pm\cos^{-1}(c_2)$$



    - ✓ The elbow-up and elbow-down posture

2.12 Inverse Kinematics Problem

- 2.12.1 Solution of Three-link Planar Arm
  - *Geometric solution technique*

$$c_\beta \sqrt{p_{Wx}^2 + p_{Wy}^2} = a_1 + a_2 c_2 \quad \rightarrow \quad \beta = \cos^{-1}\left(\frac{p_{Wx}^2 + p_{Wy}^2 + a_1^2 - a_2^2}{2a_1\sqrt{p_{Wx}^2 + p_{Wy}^2}}\right)$$

$$\alpha = \mathrm{Atan2}(p_{Wy}, p_{Wx})$$

$$\rightarrow \quad \vartheta_1 = \alpha \pm \beta$$