دانشگاه سمنان
دانشکده مهندسی مکانیک
Semnan University
Faculty of Mechanical Engineering

دانشکده مهندسی مکانیک

درس رباتیک پیشرفته

Advanced Robotics

Chapter 2 – Kinematics

Class Lecture

- ***CONTENTS:***

- Chapter 1: Introduction
- Chapter 2: **Kinematics**
- Chapter 3: Differential Kinematics and Statics
- Chapter 4: Trajectory Planning
- Chapter 5: Actuators and Sensors
- Chapter 6: Control Architecture
- Chapter 7: Dynamics
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2. Kinematics

- A manipulator:
  - Kinematic chain of rigid bodies (links) connected by means of revolute or prismatic joints.

- The derivation of the direct kinematics equation allows the end-effector position and orientation (pose) to be expressed as a function of the joint variables.

- With reference to a minimal representation of orientation, the concept of operational space is introduced and its relationship with the joint space is established.

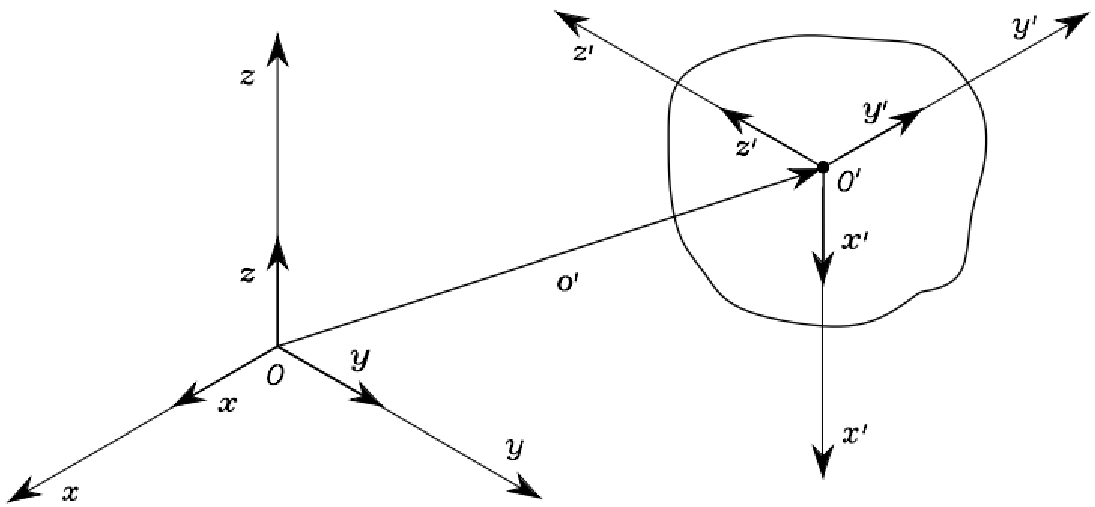
2.1 Pose of a Rigid Body

- A rigid body is completely described in space by its position and orientation (in brief pose) with respect to a reference frame.

$$\boldsymbol{o}' = o'_x \boldsymbol{x} + o'_y \boldsymbol{y} + o'_z \boldsymbol{z},$$

- Components of the vector along the frame axes

$$\boldsymbol{o}' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}$$



2.1 Pose of a Rigid Body

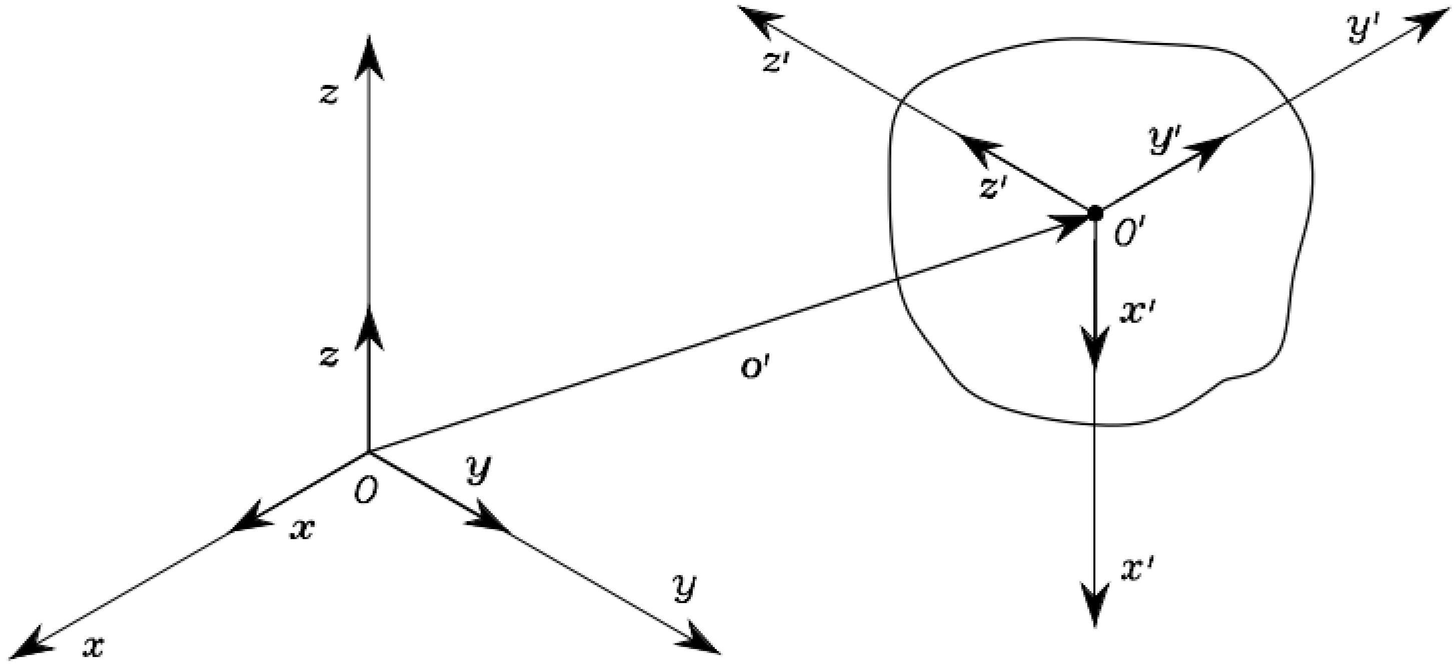
- O–xyz: Reference frame
- O’–x’y’z’: Orthonormal frame attached to the body and express its unit vectors with respect to the reference frame.

$$
\begin{aligned}
\boldsymbol{x}' &= x'_x \boldsymbol{x} + x'_y \boldsymbol{y} + x'_z \boldsymbol{z} \\
\boldsymbol{y}' &= y'_x \boldsymbol{x} + y'_y \boldsymbol{y} + y'_z \boldsymbol{z} \\
\boldsymbol{z}' &= z'_x \boldsymbol{x} + z'_y \boldsymbol{y} + z'_z \boldsymbol{z}.
\end{aligned}
$$

  - The components of each unit vector are the direction cosines of the axes of frame O’–x’y’z’ with respect to the reference frame O–xyz.

2.2 Rotation Matrix

- O–xyz and O’–x’y’z’ frames



2.2 Rotation Matrix

- Unit vectors describing body orientation with respect to reference frame

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

- Column vectors of matrix R are mutually orthogonal since they represent the unit vectors of an orthonormal frame

$$\boldsymbol{x}'^T\boldsymbol{y}' = 0 \qquad \boldsymbol{y}'^T\boldsymbol{z}' = 0 \qquad \boldsymbol{z}'^T\boldsymbol{x}' = 0.$$

- Also, they have unit norm

$$\boldsymbol{x}'^T\boldsymbol{x}' = 1 \qquad \boldsymbol{y}'^T\boldsymbol{y}' = 1 \qquad \boldsymbol{z}'^T\boldsymbol{z}' = 1.$$

2.2 Rotation Matrix

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}$$

- As a consequence, R is an orthogonal matrix

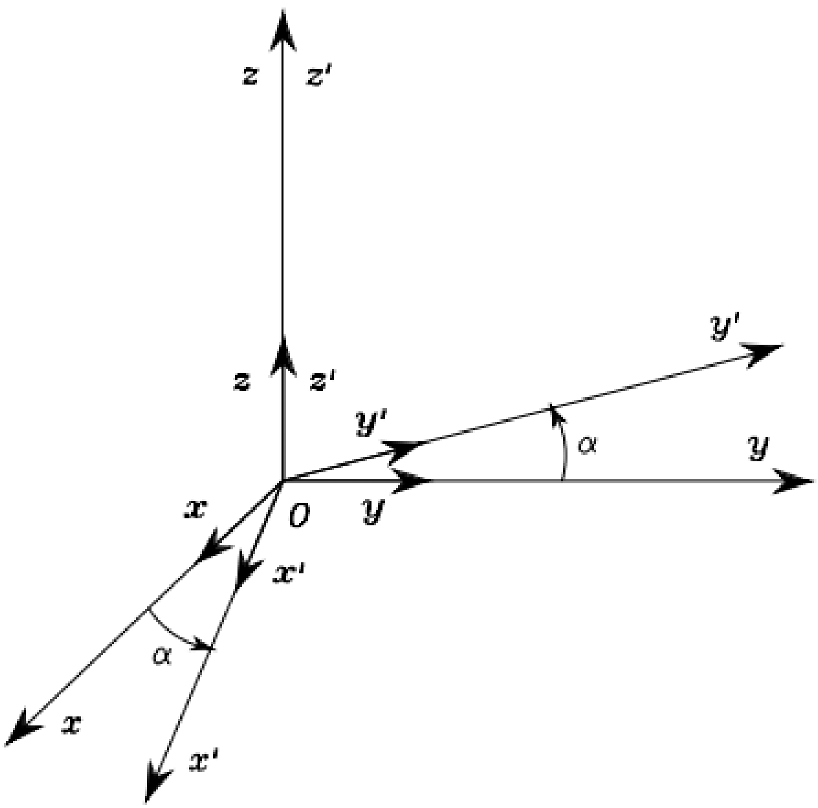
$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_3$$

$$\rightarrow \quad \boldsymbol{R}^T = \boldsymbol{R}^{-1}$$

- Right-handed frame: det(R) = 1
- Left-handed frame: det(R) = −1

2.2.1 Elementary Rotations

- Elementary rotations of the reference frame about one of the coordinate axes
  - Reference frame O–xyz is rotated by an angle α about axis z



$$x' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix}$$

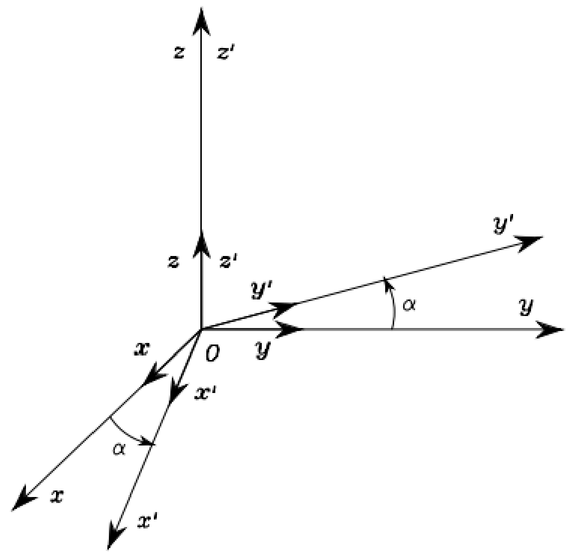
$$y' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}$$

$$z' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2.2.1 Elementary Rotations

- Rotation matrix of frame O–x’y’z’ with respect to frame O-xyz is

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



2.2.1 Elementary Rotations

- Rotations by an angle β about axis y

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

- Rotation by an angle γ about axis x

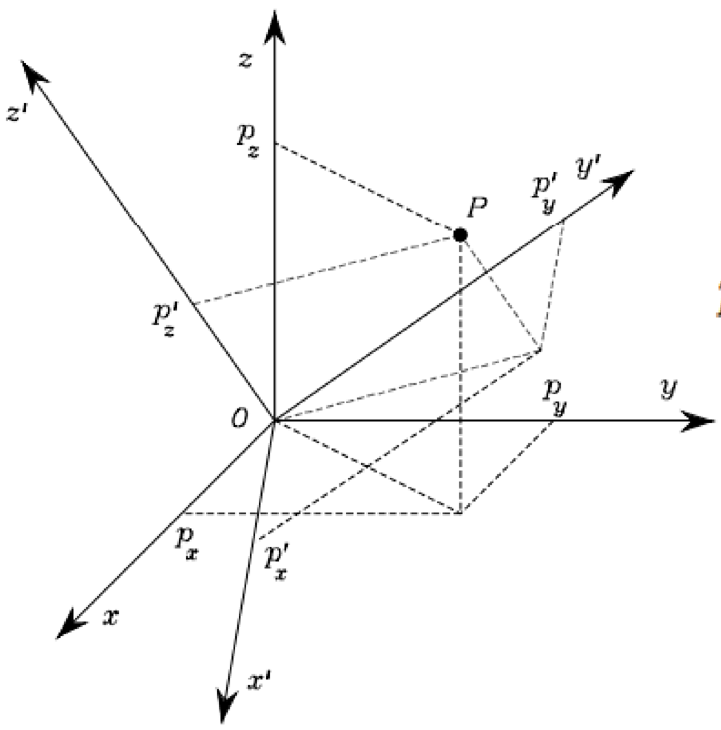
$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

- Also:

$$\rightarrow \quad \boldsymbol{R}_k(-\vartheta) = \boldsymbol{R}_k^T(\vartheta) \qquad k = x, y, z.$$

2.2.2 Representation of a Vector

- With coincident origins



$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad \boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

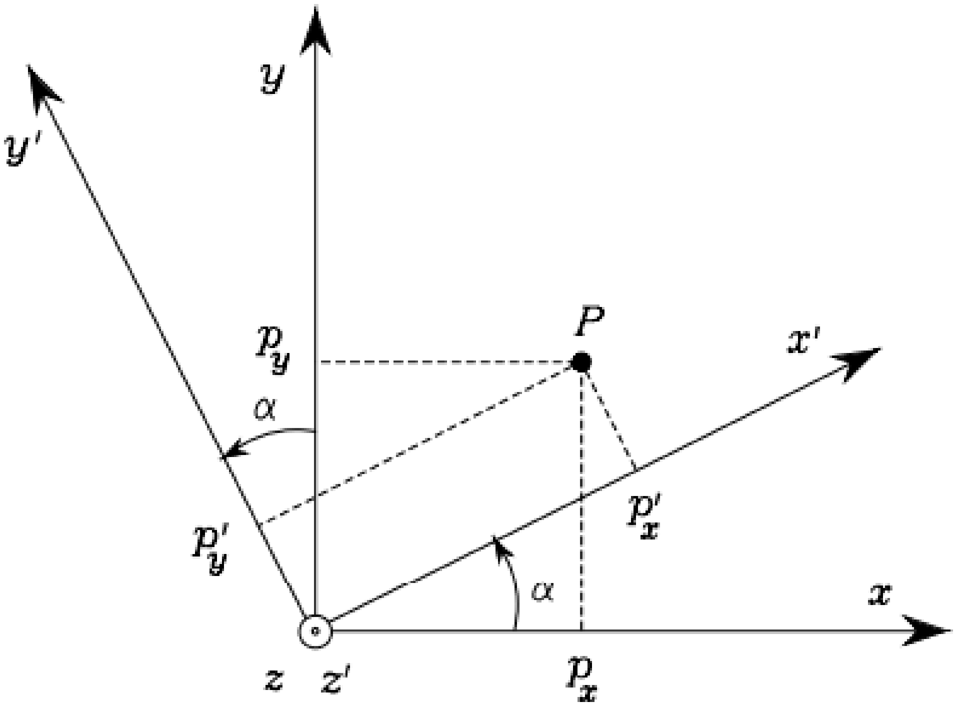
$$\boldsymbol{p} = p'_x \boldsymbol{x}' + p'_y \boldsymbol{y}' + p'_z \boldsymbol{z}' = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}'$$

$\rightarrow$ $$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'.$$

$\rightarrow$ $$\boldsymbol{p}' = \boldsymbol{R}^T \boldsymbol{p}.$$

2.2.2 Representation of a Vector

- Example 2.1



$$
\begin{aligned}
p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\
p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\
p_z &= p'_z .
\end{aligned}
$$

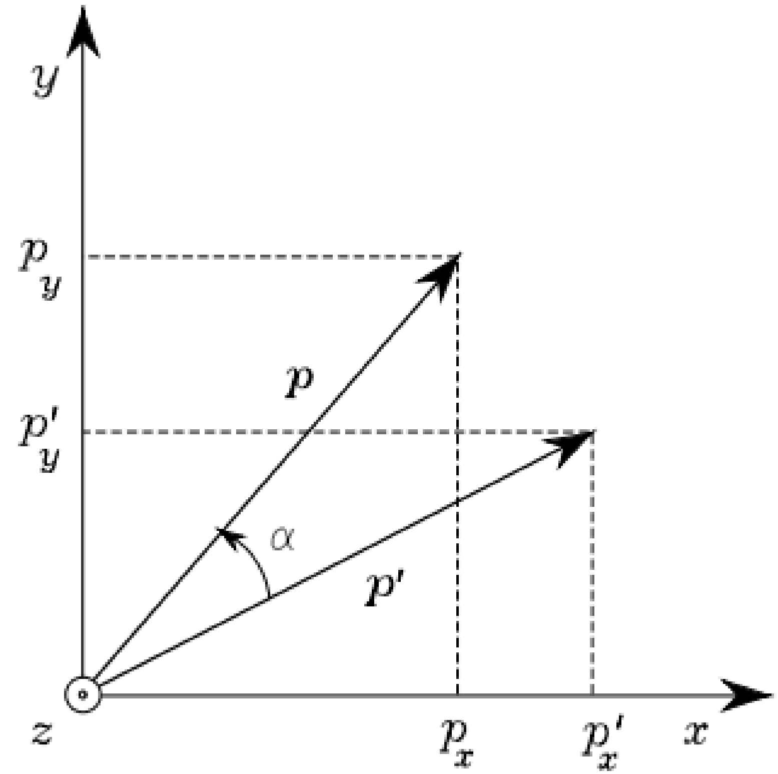
2.2.3 Rotation of a Vector

- A rotation matrix can be also interpreted as the **matrix operator** allowing rotation of a vector by a given angle about an arbitrary axis in space.

- Example 2.2

$$p_x = p'_x \cos\alpha - p'_y \sin\alpha$$
$$p_y = p'_x \sin\alpha + p'_y \cos\alpha$$
$$p_z = p'_z.$$

$$\Rightarrow \quad \boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}'$$



2.2.3 Rotation of a Vector

- A rotation matrix attains three equivalent geometrical meanings:

  - Mutual orientation between two coordinate frames

    (its column vectors are the direction cosines of the axes of the rotated frame with respect to the original frame)

  - Coordinate transformation between the coordinates of a point expressed in two different frames (with common origin)

  - An operator that allows the rotation of a vector in the same coordinate frame.

2.3 Composition of Rotation Matrices

- O–$x_0y_0z_0$, O–$x_1y_1z_1$, O–$x_2y_2z_2$ (three frames with common origin O)
- The vector p: position of a generic point in space
  - p^0, p^1, p^2 : the expressions of p in the three frames.

$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1 \boldsymbol{p}^2$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0 \boldsymbol{p}^1 \quad \rightarrow \quad \boldsymbol{R}_2^0 = \boldsymbol{R}_1^0 \boldsymbol{R}_2^1$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0 \boldsymbol{p}^2$$

  - The overall rotation can be expressed as a sequence of partial rotations

- Also:

$$\rightarrow \quad \boldsymbol{R}_i^j = (\boldsymbol{R}_j^i)^{-1} = (\boldsymbol{R}_j^i)^T$$

2.3 Composition of Rotation Matrices

- The frame with respect to which the rotation occurs is termed **current frame**.

- Composition of successive rotations is then obtained by **postmultiplication** of the rotation matrices following the given order of rotations

2.3 Composition of Rotation Matrices

- Successive rotations can be also specified by constantly referring them to the **initial frame**.
- In this case, the rotations are made with respect to a fixed frame.

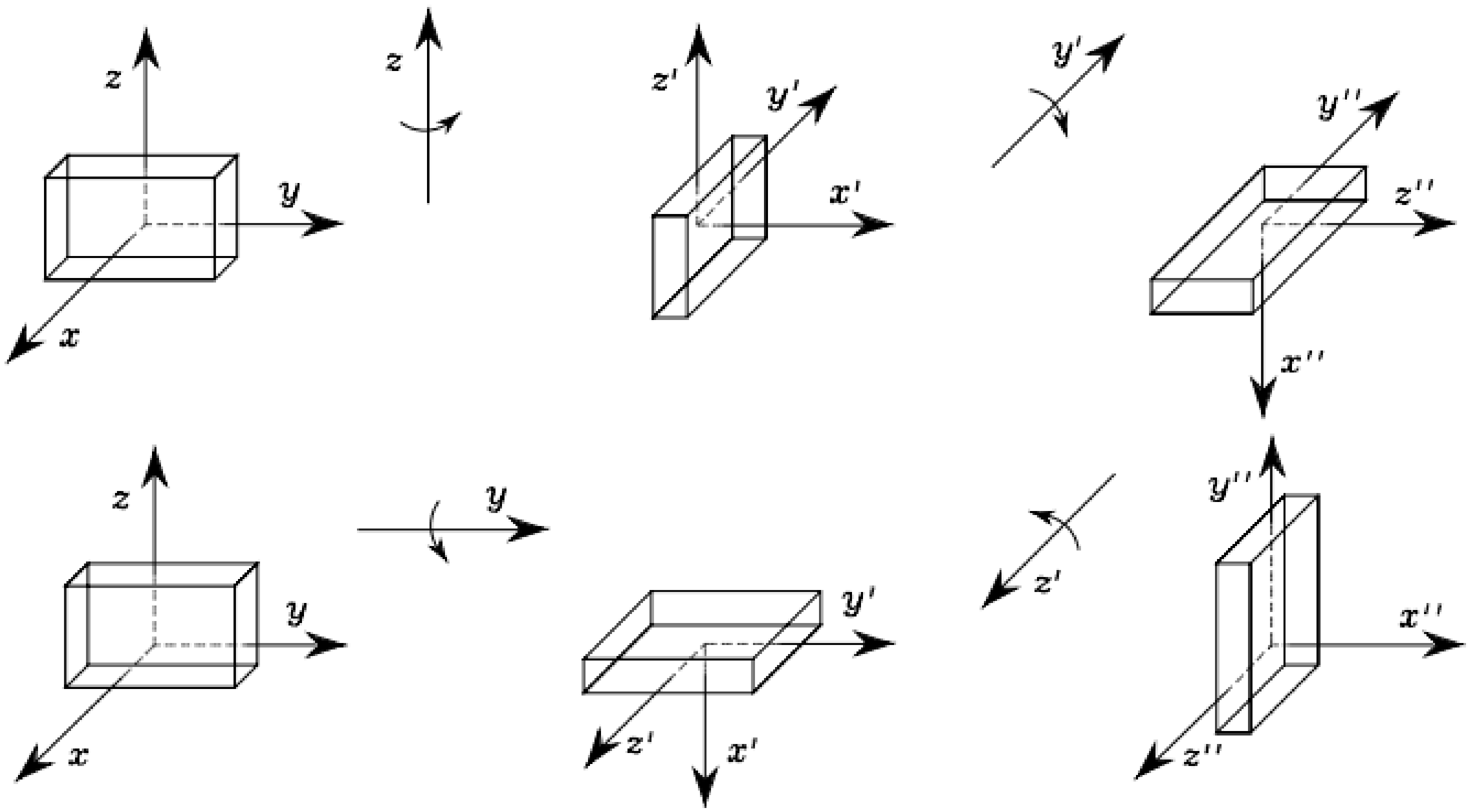
$$\bar{R}_2^0 = R_1^0 R_0^1 \bar{R}_2^1 R_1^0 \quad \rightarrow \quad \bar{R}_2^0 = \bar{R}_2^1 R_1^0$$

- Hence, it can be stated that composition of successive rotations with respect to a fixed frame is obtained by **premultiplication** of the single rotation matrices in the order of the given sequence of rotations.

  - An important issue of composition of rotations is that the matrix product is not commutative.

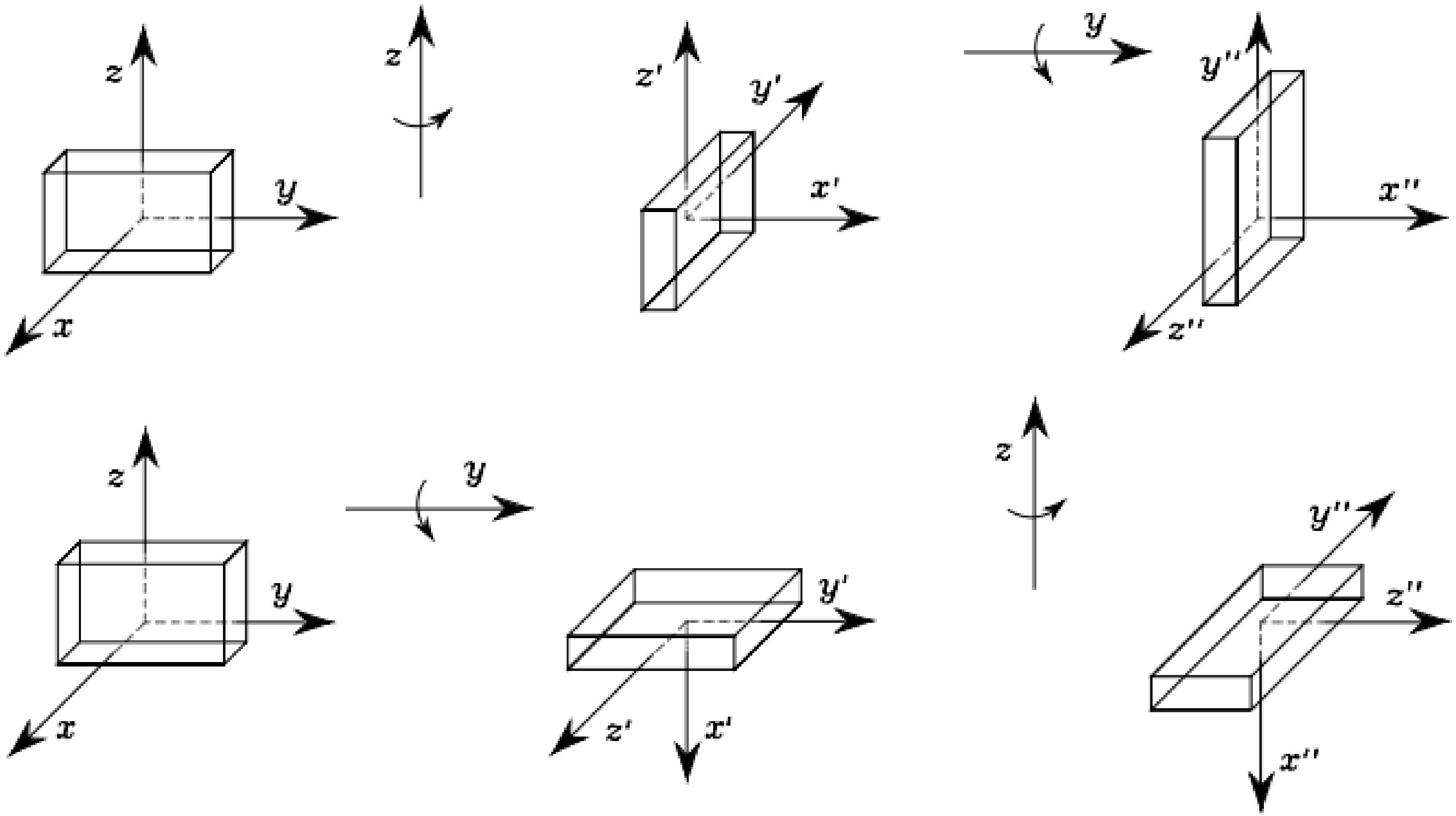
2.3 Composition of Rotation Matrices

- Example 2.3
  - Successive rotations of an object about axes of current frame



2.3 Composition of Rotation Matrices

- Example 2.3
  - Successive rotations of an object about axes of fixed frame



2.4 Euler Angles

- Rotation matrices give a redundant description of frame orientation; in fact, they are characterized by nine elements which are not independent but related by six constraints due to the orthogonality conditions.

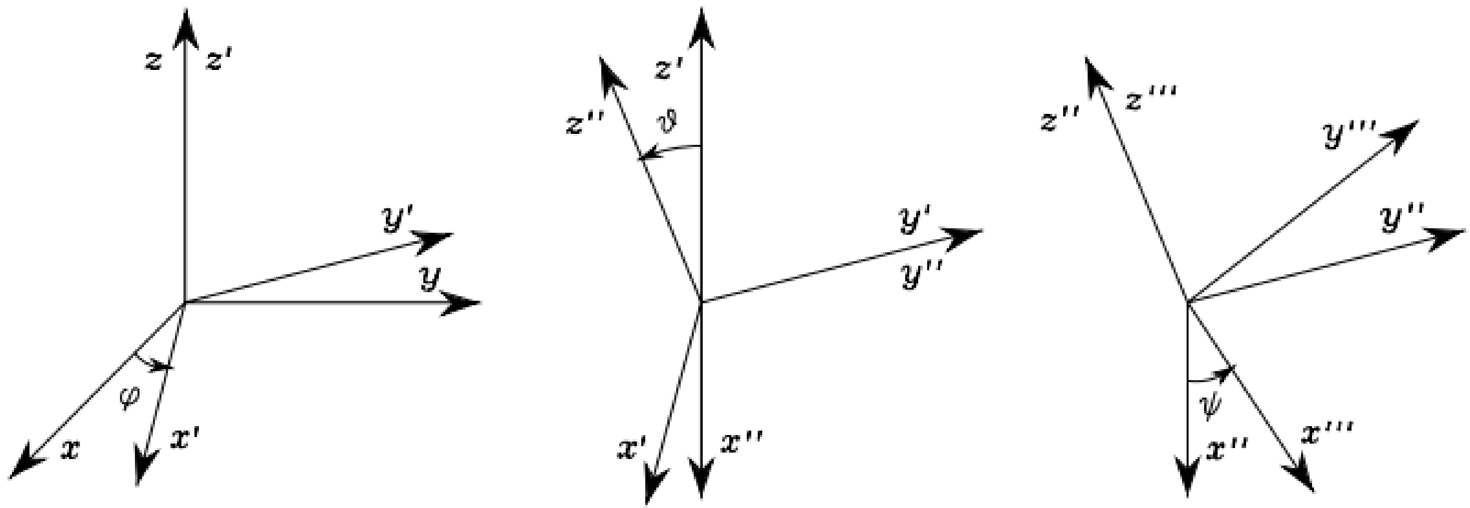
- 3 parameters are sufficient to describe orientation of a rigid body in space. (minimal representation)

- A minimal representation of orientation can be obtained by using a set of three angles $[\,\phi \;\; \vartheta \;\; \psi\,]^T$.

- 12 distinct sets of angles are allowed out of all 27 possible combinations; each set represents a triplet of Euler angles.

2.4 Euler Angles

- 2.4.1 ZYZ Angles
  - Rotate the reference frame by the angle ϕ about axis z
  - Rotate the current frame by the angle ϑ about axis y
  - Rotate the current frame by the angle ψ about axis z



2.4 Euler Angles

- 2.4.1 ZYZ Angles
  - The resulting frame orientation is obtained by composition of rotations with respect to current frames

$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)$$
$$= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}$$

2.4 Euler Angles

- 2.4.1 ZYZ Angles
  - The inverse problem

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

ϑ to $(0, \pi)$ →

$$\varphi = \text{Atan2}(r_{23}, r_{13})$$

$$\vartheta = \text{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \text{Atan2}(r_{32}, -r_{31}).$$

2.4 Euler Angles

- 2.4.1 ZYZ Angles
  - The inverse problem

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

ϑ in the range $(-\pi, 0)$ →

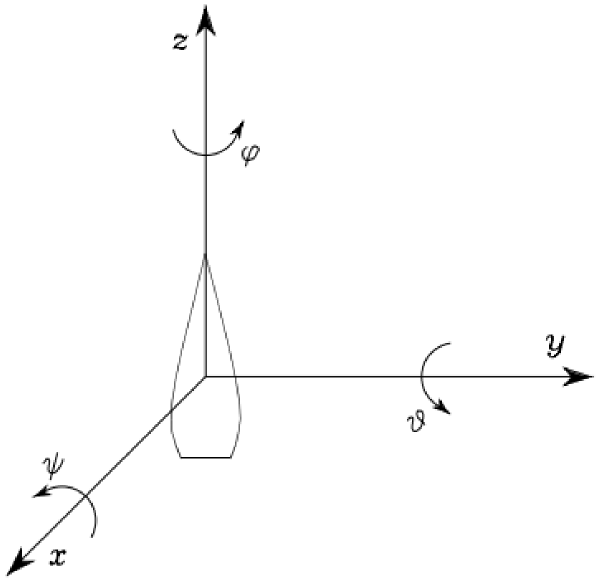
$$\varphi = \text{Atan2}(-r_{23}, -r_{13})$$

$$\vartheta = \text{Atan2}\left(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)$$

$$\psi = \text{Atan2}(-r_{32}, r_{31}).$$

2.4 Euler Angles

- 2.4.2 RPY Angles
  - Representation of orientation in the aeronautical field.
  - These are the ZYX angles, also called Roll–Pitch–Yaw angles, to denote the typical changes of attitude of an aircraft.
  - The angles $[\ \phi \quad \vartheta \quad \psi\ ]^T$ represent rotations defined with respect to a fixed frame attached to the center of mass of the aircraft.



2.4 EULER ANGLES

- 2.4.2 RPY Angles
  - Rotate the reference frame by the angle ψ about axis x (yaw)
  - Rotate the reference frame by the angle ϑ about axis y (pitch)
  - Rotate the reference frame by the angle ϕ about axis z (roll)

$$\boldsymbol{R}(\boldsymbol{\phi}) = \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi)$$

$$= \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}$$

2.4 Euler Angles

- 2.4.2 RPY Angles
  - The inverse solution

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\varphi = \text{Atan2}(r_{21}, r_{11})$$

ϑ in the range $(-\pi/2, \pi/2)$ →

$$\vartheta = \text{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right)$$

$$\psi = \text{Atan2}(r_{32}, r_{33}).$$

2.4 Euler Angles

- 2.4.2 RPY Angles
  - The inverse solution

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\varphi = \text{Atan2}(-r_{21}, -r_{11})$$

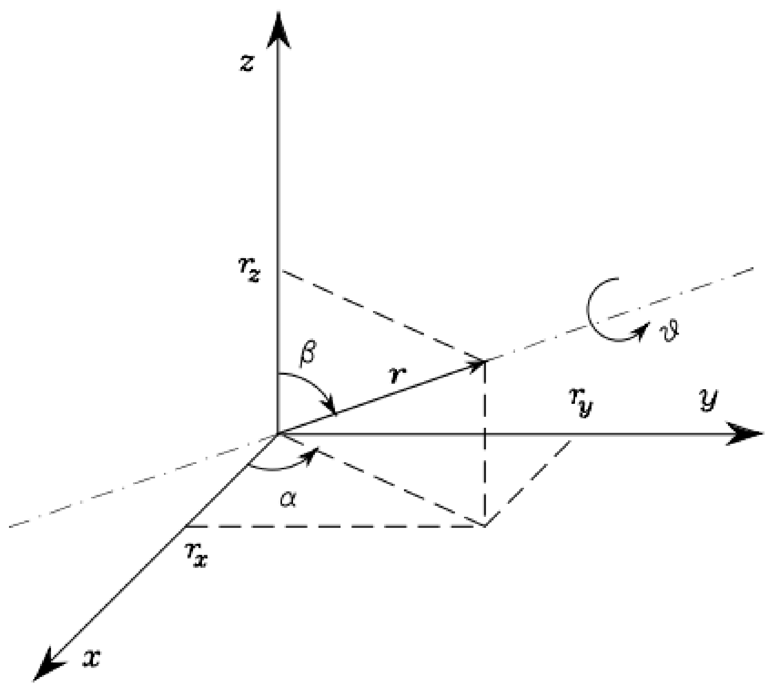
ϑ in the range $(\pi/2, 3\pi/2)$ →

$$\vartheta = \text{Atan2}\left(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}\right)$$

$$\psi = \text{Atan2}(-r_{32}, -r_{33}).$$

2.5 Angle and Axis

- A nonminimal representation: rotation of a given angle about an axis in space (with 4 parameters)
- This can be advantageous in the problem of trajectory planning for a manipulator's end-effector orientation.



2.5 Angle and Axis

- Let $r = [\, r_x \; r_y \; r_z \,]^T$ be the unit vector of a rotation axis with respect to the reference frame O–xyz.
- In order to derive the rotation matrix $R(\vartheta, r)$ expressing the rotation of an angle ϑ about axis r

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha)$$

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z.$$

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \begin{bmatrix} r_x^2(1-c_\vartheta) + c_\vartheta & r_x r_y(1-c_\vartheta) - r_z s_\vartheta & r_x r_z(1-c_\vartheta) + r_y s_\vartheta \\ r_x r_y(1-c_\vartheta) + r_z s_\vartheta & r_y^2(1-c_\vartheta) + c_\vartheta & r_y r_z(1-c_\vartheta) - r_x s_\vartheta \\ r_x r_z(1-c_\vartheta) - r_y s_\vartheta & r_y r_z(1-c_\vartheta) + r_x s_\vartheta & r_z^2(1-c_\vartheta) + c_\vartheta \end{bmatrix}$$

2.5 Angle and Axis

- The inverse problem

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\vartheta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$

$$\boldsymbol{r} = \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix},$$

2.5 Angle and Axis

- The three components of r are not independent but are constrained:

$$r_x^2 + r_y^2 + r_z^2 = 1$$

- If sin ϑ = 0, the expressions become meaningless. To solve the inverse problem, it is necessary to directly refer to the particular expressions attained by the rotation matrix R and find the solving formulae in the two cases ϑ = 0 and ϑ = π .

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2.6 Unit Quaternion

- The drawbacks of the angle/axis representation can be overcome by a different four-parameter representation; namely, the **Unit Quaternion**

$$\mathcal{Q} = \{\eta, \boldsymbol{\epsilon}\}$$

$$\eta = \cos\frac{\vartheta}{2}$$

→

$$\boldsymbol{\epsilon} = \sin\frac{\vartheta}{2}\boldsymbol{r}$$

2.6 Unit Quaternion

- η : the scalar part of the quaternion while
- $\varepsilon = [\varepsilon_x \; \varepsilon_y \; \varepsilon_z]^T$: the vector part of the quaternion.

$$\eta^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 = 1$$

$$\boldsymbol{R}(\eta, \boldsymbol{\epsilon}) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}$$

2.6 Unit Quaternion

- The inverse problem

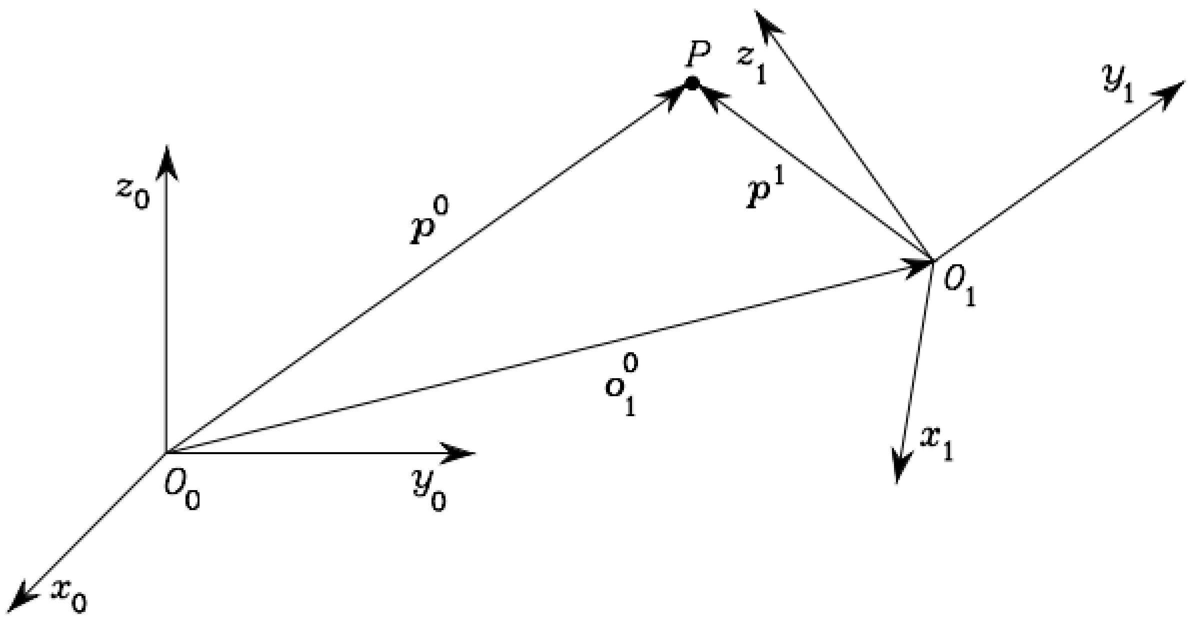
$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\eta = \frac{1}{2}\sqrt{r_{11} + r_{22} + r_{33} + 1}$$

$$\epsilon = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}$$

2.7 Homogeneous Transformations

- The position of a rigid body in space:
  - Position of a point on the body with respect to a reference frame (translation)
  - Components of the unit vectors (orientation) of a frame attached to the body with respect to the same reference frame (rotation)



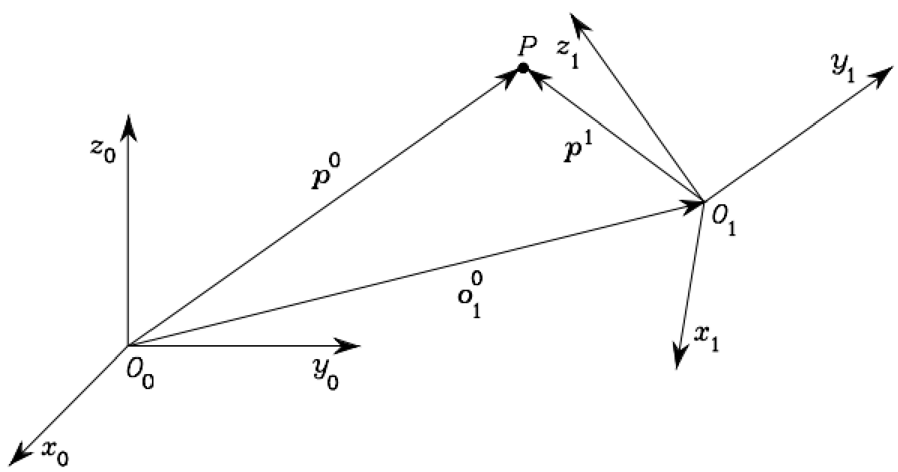
2.7 Homogeneous Transformations

- Coordinate transformation (translation + rotation) of a bound vector between two frames:
  - ✓ R_0^1 : Rotation matrix of Frame 1 with respect to Frame 0

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

- The inverse transformation:

$$\boldsymbol{p}^1 = -\boldsymbol{R}_1^{0T} \boldsymbol{o}_1^0 + \boldsymbol{R}_1^{0T} \boldsymbol{p}^0 \quad \longrightarrow \quad \boldsymbol{p}^1 = -\boldsymbol{R}_0^1 \boldsymbol{o}_1^0 + \boldsymbol{R}_0^1 \boldsymbol{p}^0$$



2.7 Homogeneous Transformations

- The **Homogeneous Representation** of a generic vector **p** : (**p**˜)
  - In order to achieve a compact representation of the relationship between the coordinates of the same point in two different frames

$$\widetilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix} \rightarrow \begin{aligned} \widetilde{\boldsymbol{p}}^0 &= \boldsymbol{A}_1^0 \widetilde{\boldsymbol{p}}^1 \\ \widetilde{\boldsymbol{p}}^1 &= \boldsymbol{A}_0^1 \widetilde{\boldsymbol{p}}^0 = \left(\boldsymbol{A}_1^0\right)^{-1} \widetilde{\boldsymbol{p}}^0 \end{aligned}$$

  - *Homogeneous Transformation Matrix*

$$\boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

2.7 Homogeneous Transformations

- *Homogeneous Transformation Matrix*

$$A_1^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0^T & 1 \end{bmatrix}$$

$$A_0^1 = \begin{bmatrix} R_1^{0T} & -R_1^{0T} o_1^0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_0^1 & -R_0^1 o_1^0 \\ 0^T & 1 \end{bmatrix}$$

- Notice that:

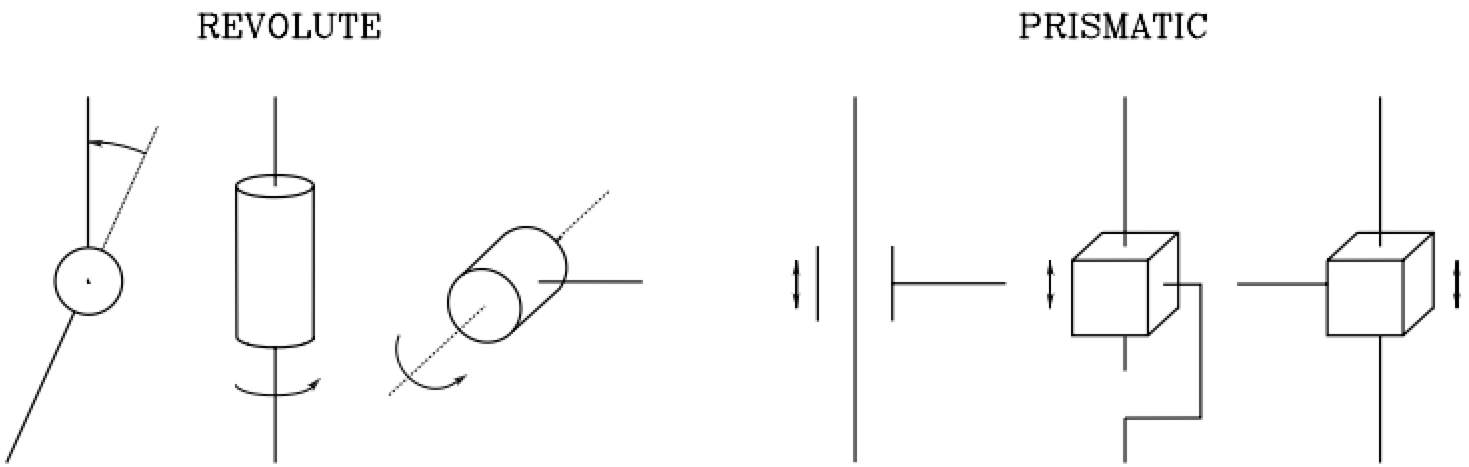
$$A^{-1} \neq A^T$$

$$\tilde{p}^0 = A_1^0 A_2^1 \ldots A_n^{n-1} \tilde{p}^n$$

2.8 DIRECT KINEMATICS

- A manipulator:
  - Series of rigid bodies (links) connected by means of kinematic pairs or joints

- Joints:
  - Revolute
  - Prismatic



2.8 Direct Kinematics

- The whole structure forms a Kinematic Chain.
  - ✓ One end of the chain is constrained to a base.
  - ✓ The other end is an end-effector (gripper, tool)

- Open kinematic chain (only one sequence of links connecting the two ends)
- Closed kinematic chain (a sequence of links forms a loop)

- Characterized by a number of degrees of freedom (DOFs)
  - ✓ Uniquely determine its posture.
  - ✓ Each DOF is typically associated with a joint articulation and constitutes a joint variable

Direct kinematics:

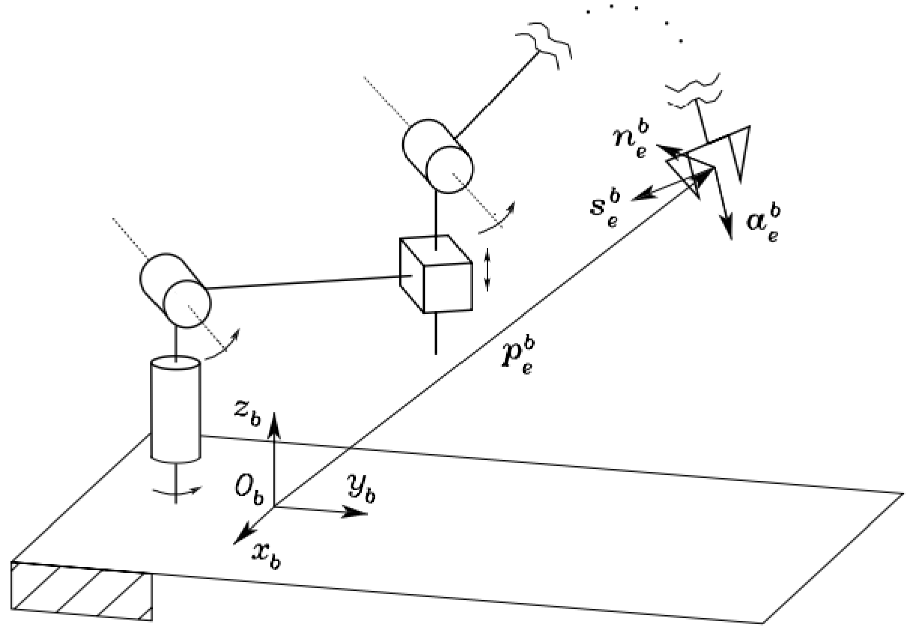
- Compute the pose of the end-effector as a function of the joint variables

2.8 Direct Kinematics

- Direct kinematics function homogeneous transformation matrix

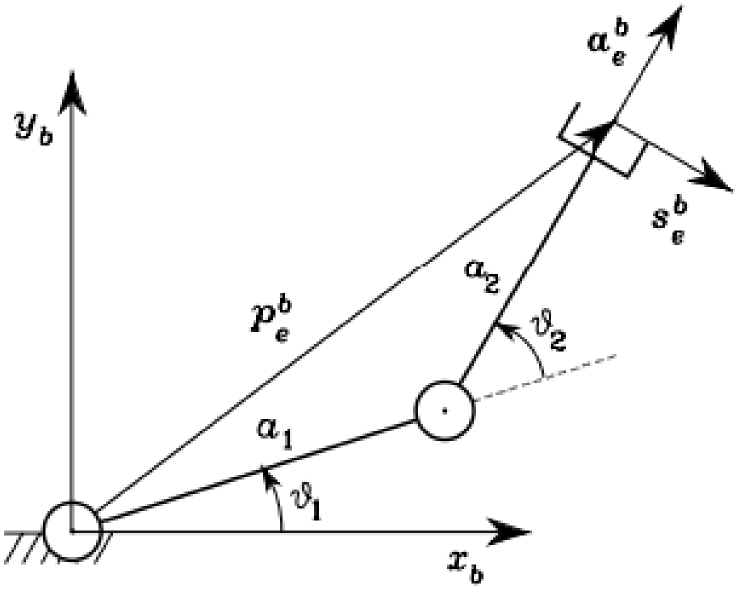
$$T_e^b(q) = \begin{bmatrix} n_e^b(q) & s_e^b(q) & a_e^b(q) & p_e^b(q) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - n_e, s_e, a_e and p_e are a function of q



2.8 Direct Kinematics

- Example 2.4
  - Two-link planar arm

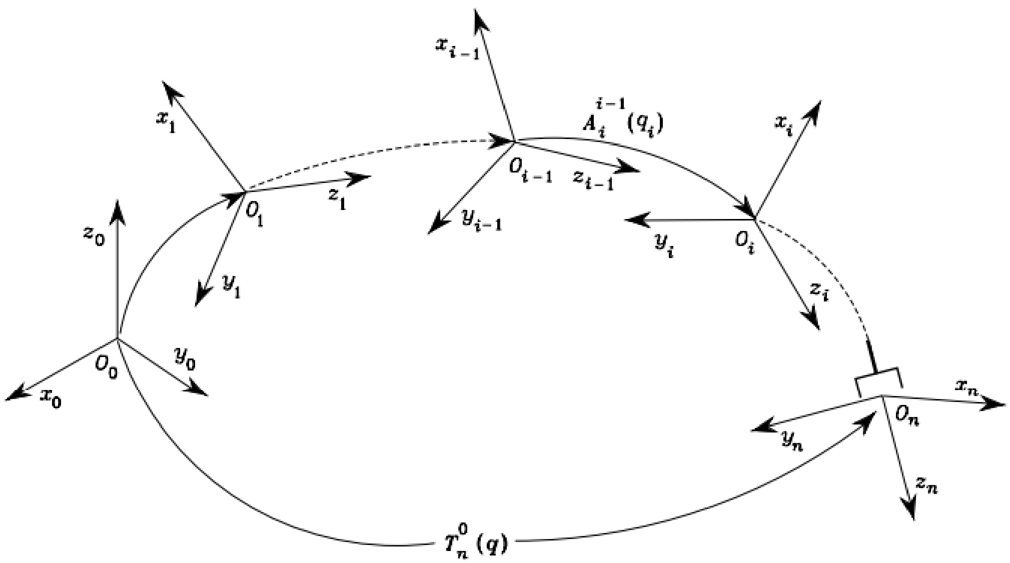


$$T_e^b(q) = \begin{bmatrix} n_e^b & s_e^b & a_e^b & p_e^b \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & s_{12} & c_{12} & a_1c_1 + a_2c_{12} \\ 0 & -c_{12} & s_{12} & a_1s_1 + a_2s_{12} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.8 Direct Kinematics

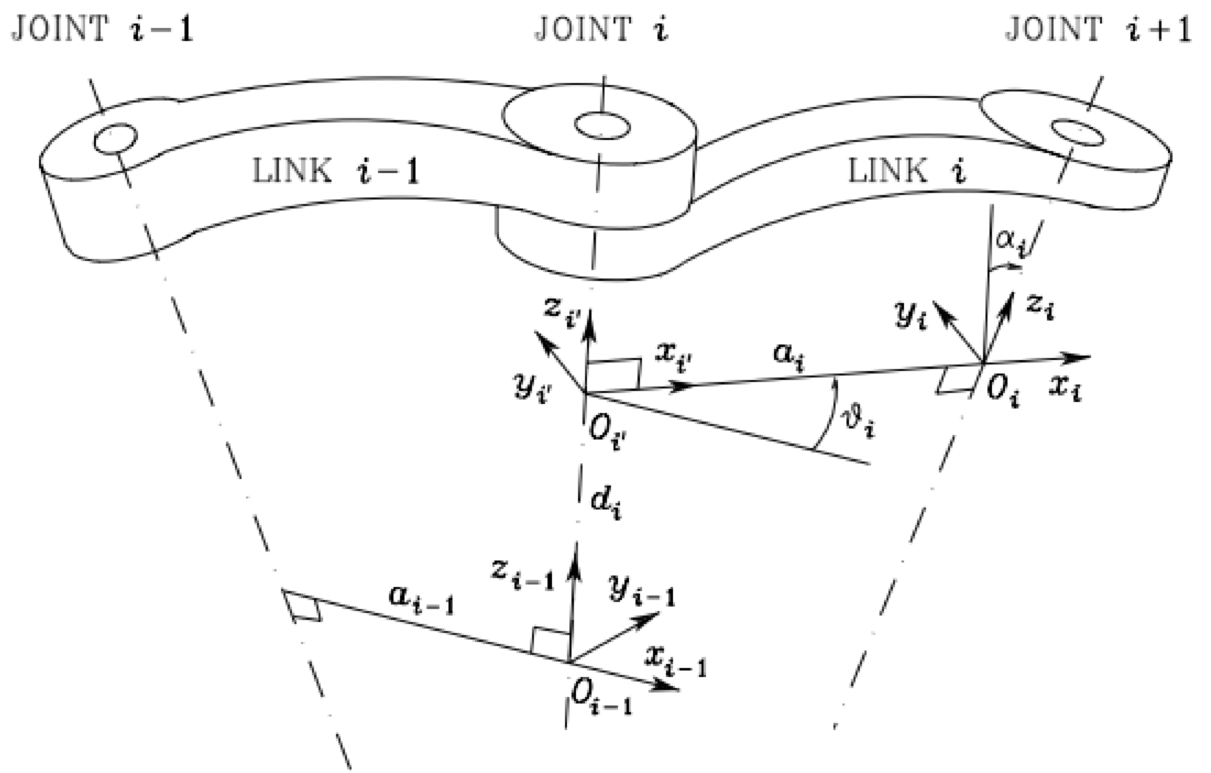
- 2.8.1 Open Chain
  - An open-chain manipulator constituted by n + 1 links connected by n joints
  - Define a coordinate frame attached to each link, from Link 0 to Link n
  - The coordinate transformation describing the position and orientation of Frame n with respect to Frame 0:

$$\boldsymbol{T}_n^0(\boldsymbol{q}) = \boldsymbol{A}_1^0(q_1)\boldsymbol{A}_2^1(q_2)\dots\boldsymbol{A}_n^{n-1}(q_n) \longrightarrow \boldsymbol{T}_e^b(\boldsymbol{q}) = \boldsymbol{T}_0^b\boldsymbol{T}_n^0(\boldsymbol{q})\boldsymbol{T}_e^n$$



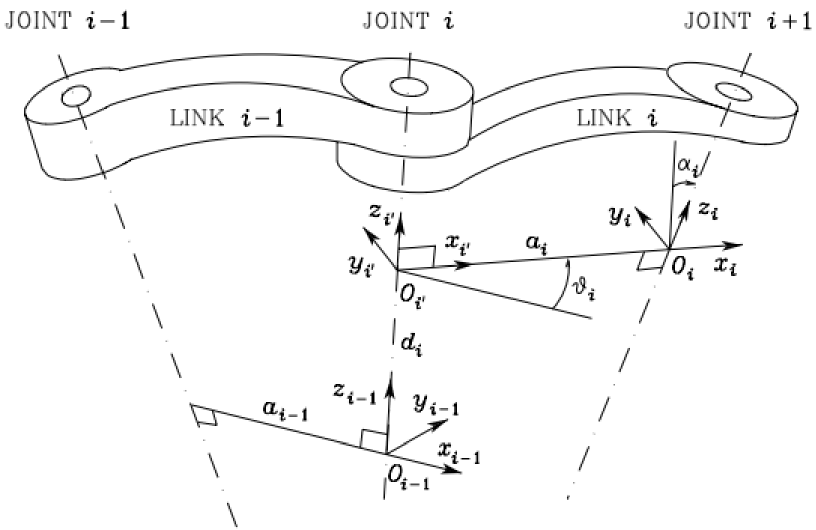
2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - A systematic, general method is to be derived to define the relative position and orientation of two consecutive links



2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Let Axis i denote the axis of the joint connecting Link i − 1 to Link i
  - The Denavit–Hartenberg convention (DH) is adopted to define link Frame i:
    - ✓ Choose axis zi along the axis of Joint i + 1
    - ✓ Locate the origin O_i and O_i’
    - ✓ Choose axis x_i along the common normal to axes z_{i-1} and z_i (from Joint i to Joint i + 1)
    - ✓ Choose axis y_i so as to complete a right-handed frame.



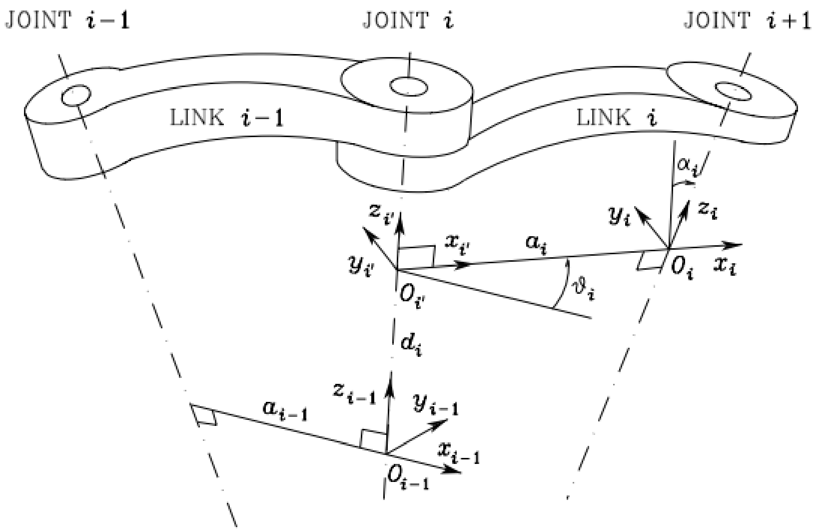
2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - The Denavit–Hartenberg convention gives a nonunique definition of the link frame in the following cases:
    - ✓ For Frame 0, only the direction of axis z_0 is specified; O_0 and x_0 can be arbitrarily chosen
    - ✓ For Frame n (no Joint n+1) z_n is not uniquely defined while x_n has to be normal to axis z_{n-1}

      (Typically, Joint n is revolute, and thus z_n is to be aligned with the direction of z_{n-1})
    - ✓ When two consecutive axes are parallel, the common normal is not uniquely defined
    - ✓ When two consecutive axes intersect, the direction of xi is arbitrary
    - ✓ When Joint i is prismatic, the direction of z_{i-1} is arbitrary

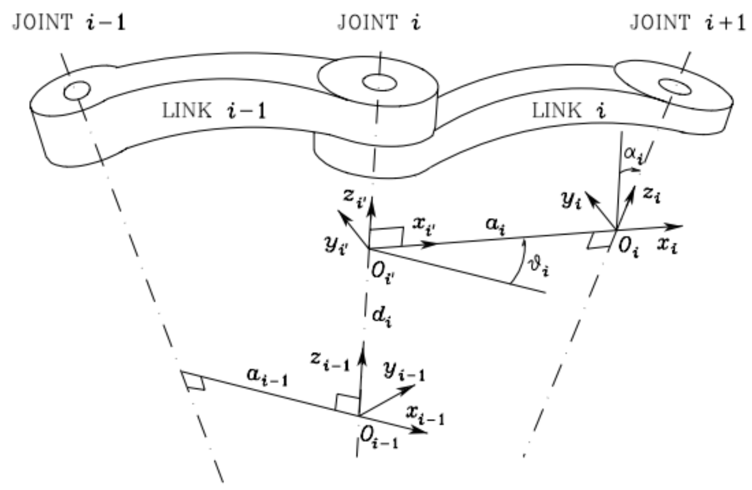
2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Parameters:
    - ✓ a_i : Distance between O_i and O_i'
    - ✓ d_i : Coordinate of O_i along z_{i-1}
    - ✓ αi : Angle between axes z_{i-1} and z_i about axis xi (positive: counter-clockwise)
    - ✓ ϑi : Angle between axes x_{i-1} and x_i about axis z_{i-1} (positive: counter-clockwise)



2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Two of the four parameters (a_i and α_i) are always constant and depend only on the geometry of connection between consecutive joints.
  - ***If Joint i is revolute the variable is ϑ_i***
  - ***If Joint i is prismatic the variable is d_i***



2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Coordinate transformation between Frame i and Frame i − 1:
    1. Choose a frame aligned with Frame i − 1
    2. Translate the chosen frame by d_i along axis z_{i-1} and rotate it by ϑ_i about axis z_{i-1}

$$\boldsymbol{A}_{i'}^{i-1} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} & 0 & 0 \\ s_{\vartheta_i} & c_{\vartheta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.8 Direct Kinematics

- 2.8.2 Denavit–Hartenberg Convention
  - Coordinate transformation between Frame i and Frame i − 1:

3. Translate the frame aligned with Frame i' by a_i along x_i and rotate it by α_i about x_i

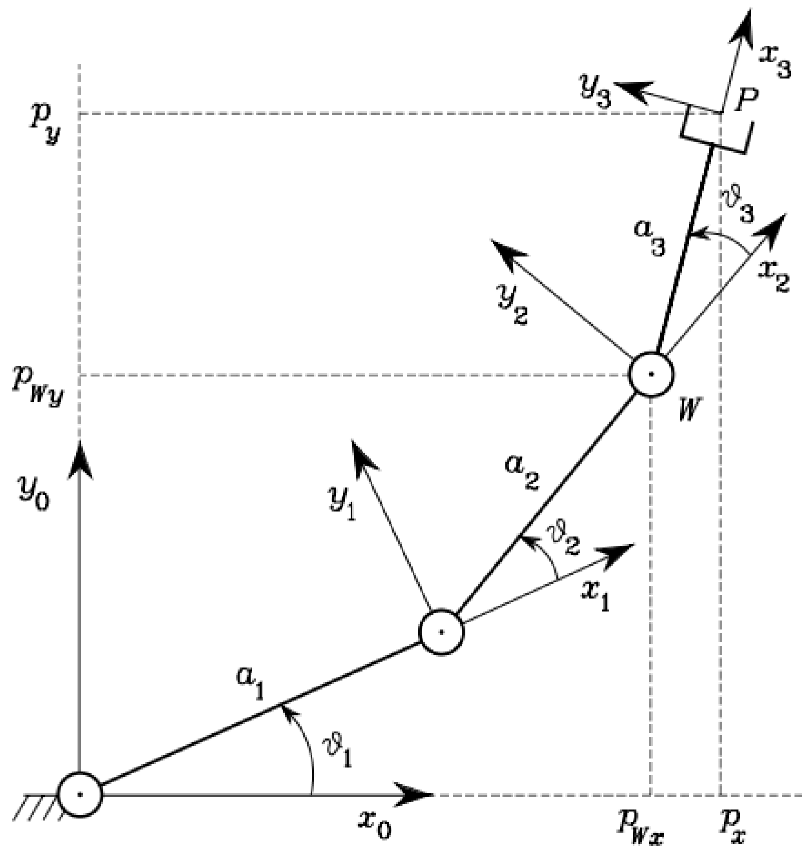
$$A_i^{i'} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Post-multiplicate the single transformations:

$$A_i^{i-1}(q_i) = A_{i'}^{i-1} A_i^{i'} = \begin{bmatrix} c_{\vartheta_i} & -s_{\vartheta_i} c_{\alpha_i} & s_{\vartheta_i} s_{\alpha_i} & a_i c_{\vartheta_i} \\ s_{\vartheta_i} & c_{\vartheta_i} c_{\alpha_i} & -c_{\vartheta_i} s_{\alpha_i} & a_i s_{\vartheta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.9 Kinematics of Typical Manipulator Structures

- 2.9.1 Three-link Planar Arm

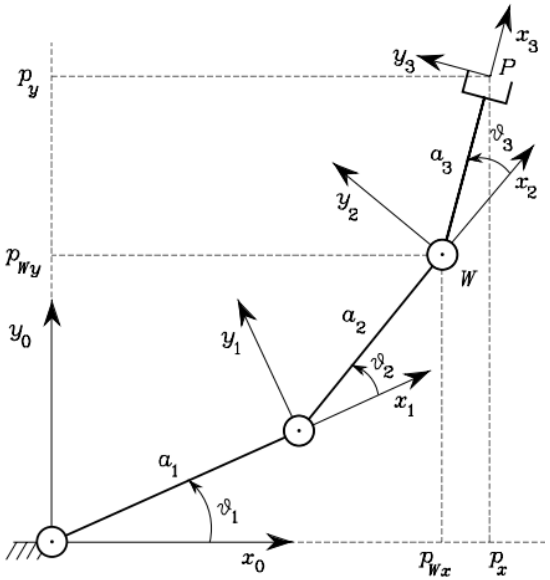


2.9 Kinematics of Typical Manipulator Structures

- 2.9.1 Three-link Planar Arm
  - DH Parameters

Link	a_i	α_i	d_i	ϑ_i
1	a_1	0	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3

$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, 2, 3$$



2.9 Kinematics of Typical Manipulator Structures

- 2.9.1 Three-link Planar Arm
  - All joints are revolute:

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ s_{123} & c_{123} & 0 & a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

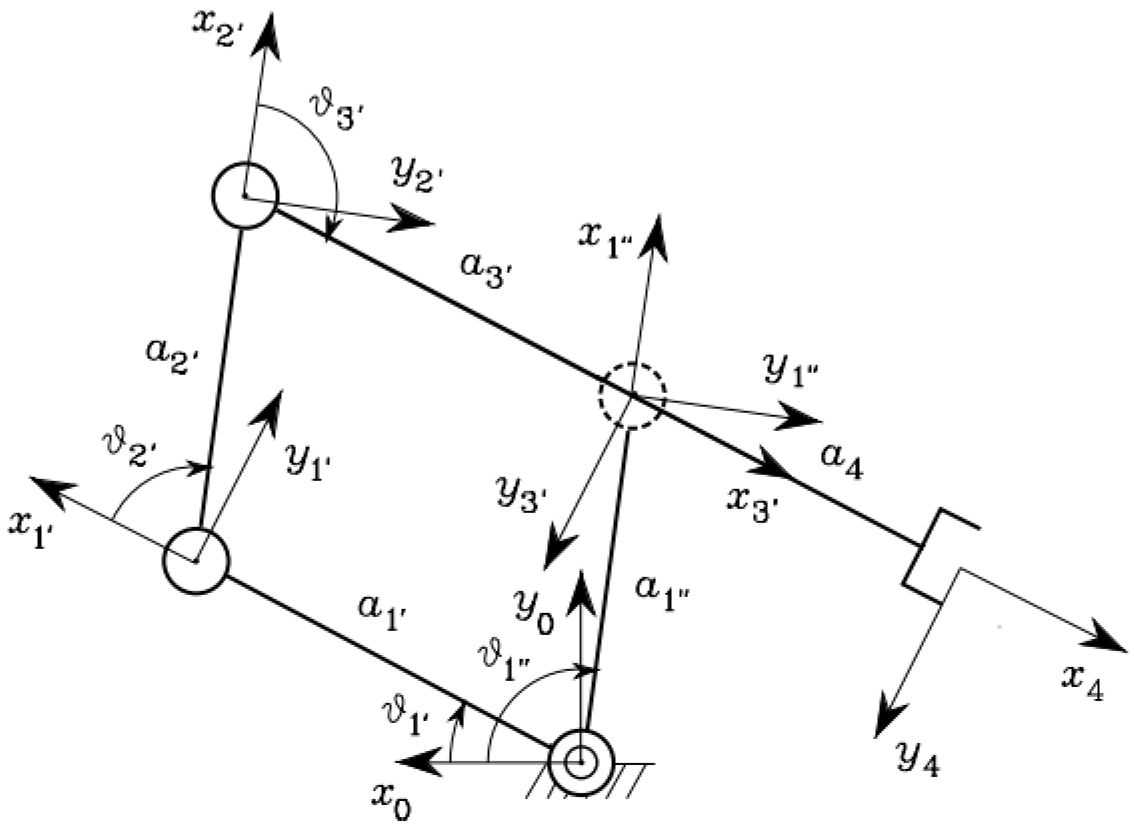
$$\boldsymbol{q} = \begin{bmatrix} \vartheta_1 & \vartheta_2 & \vartheta_3 \end{bmatrix}^T$$

  - End-effector frame:

$$\boldsymbol{T}_e^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.9 Kinematics of Typical Manipulator Structures

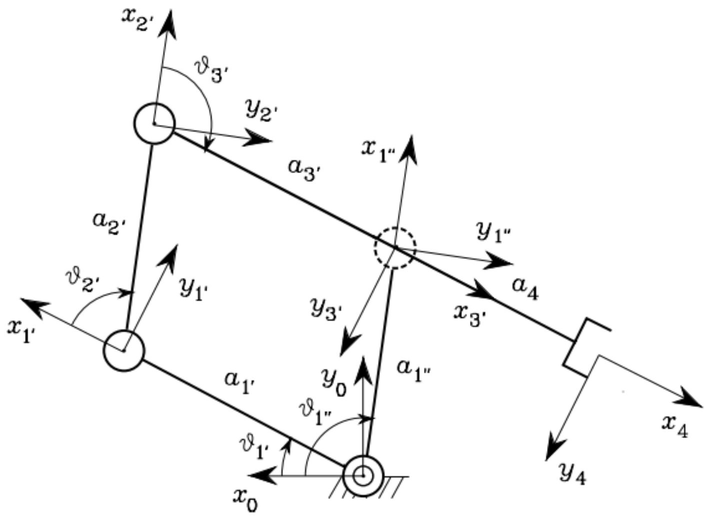
- 2.9.2 Parallelogram Arm



2.9 Kinematics of Typical Manipulator Structures

- 2.9.2 Parallelogram Arm
  - DH Parameters

Link	a_i	α_i	d_i	ϑ_i
$1'$	$a_{1'}$	0	0	$\vartheta_{1'}$
$2'$	$a_{2'}$	0	0	$\vartheta_{2'}$
$3'$	$a_{3'}$	0	0	$\vartheta_{3'}$
$1''$	$a_{1''}$	0	0	$\vartheta_{1''}$
4	a_4	0	0	0



2.9 Kinematics of Typical Manipulator Structures

- 2.9.2 Parallelogram Arm
  - The coordinate transformations for the two branches:

$$\boldsymbol{A}_{3'}^{0}(\boldsymbol{q}') = \boldsymbol{A}_{1'}^{0}\boldsymbol{A}_{2'}^{1'}\boldsymbol{A}_{3'}^{2'} = \begin{bmatrix} c_{1'2'3'} & -s_{1'2'3'} & 0 & a_{1'}c_{1'} + a_{2'}c_{1'2'} + a_{3'}c_{1'2'3'} \\ s_{1'2'3'} & c_{1'2'3'} & 0 & a_{1'}s_{1'} + a_{2'}s_{1'2'} + a_{3'}s_{1'2'3'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{q}' = [\,\vartheta_{1'} \quad \vartheta_{2'} \quad \vartheta_{3'}\,]^T$$

$$\boldsymbol{A}_{1''}^{0}(q'') = \begin{bmatrix} c_{1''} & -s_{1''} & 0 & a_{1''}c_{1''} \\ s_{1''} & c_{1''} & 0 & a_{1''}s_{1''} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad q'' = \vartheta_{1''}$$

2.9 Kinematics of Typical Manipulator Structures

- 2.9.2 Parallelogram Arm
  - The constant homogeneous transformation for the last link:

$$\boldsymbol{A}_4^{3'} = \begin{bmatrix} 1 & 0 & 0 & a_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

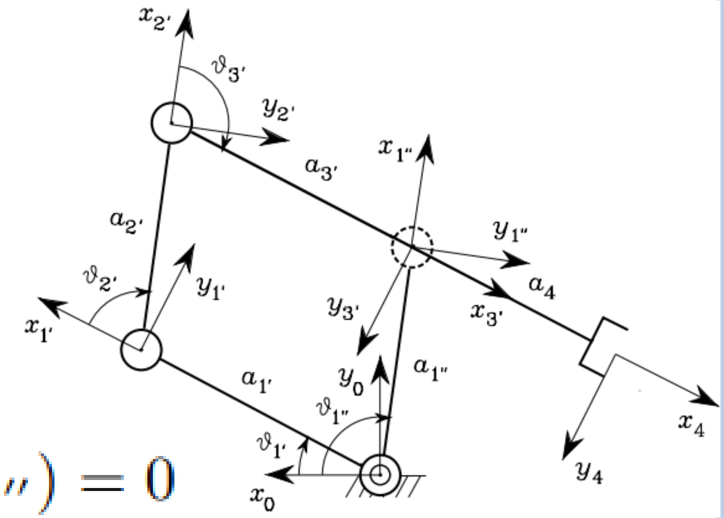
  - The position constraints:

$$d_{3'1''} = 0$$

$$\boldsymbol{R}_0^{3'}(\boldsymbol{q}')\left(\boldsymbol{p}_{3'}^0(\boldsymbol{q}') - \boldsymbol{p}_{1''}^0(q'')\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2.9 Kinematics of Typical Manipulator Structures

- 2.9.2 Parallelogram Arm



$$a_{1'} = a_{3'} \text{ and } a_{2'} = a_{1''}$$

$$a_{1'}(c_{1'} + c_{1'2'3'}) + a_{1''}(c_{1'2'} - c_{1''}) = 0$$
$$a_{1'}(s_{1'} + s_{1'2'3'}) + a_{1''}(s_{1'2'} - s_{1''}) = 0$$

$$\vartheta_{2'} = \vartheta_{1''} - \vartheta_{1'}$$
$$\vartheta_{3'} = \pi - \vartheta_{2'} = \pi - \vartheta_{1''} + \vartheta_{1'}$$

2.9 Kinematics of Typical Manipulator Structures

- 2.9.2 Parallelogram Arm
  - The vector of joint variables:

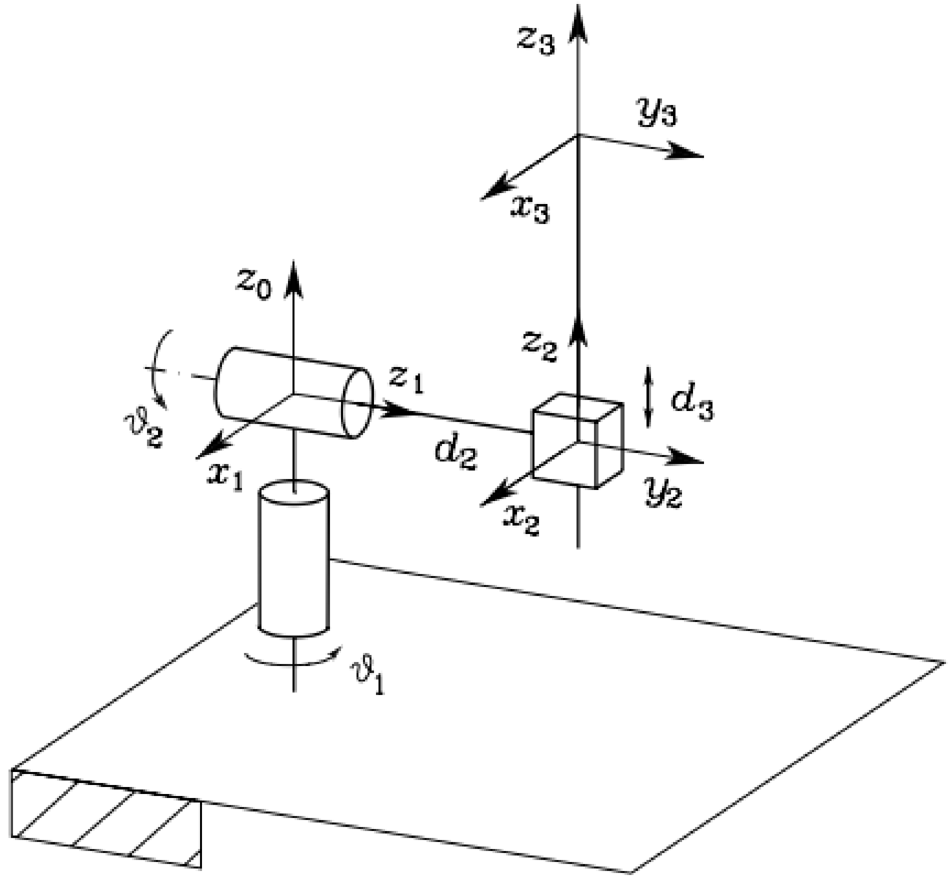
$$\boldsymbol{q} = \begin{bmatrix} \vartheta_{1'} & \vartheta_{1''} \end{bmatrix}^T$$

  - Direct Kinematic Function:

$$\boldsymbol{T}_4^0(\boldsymbol{q}) = \boldsymbol{A}_{3'}^0(\boldsymbol{q})\boldsymbol{A}_4^{3'} = \begin{bmatrix} -c_{1'} & s_{1'} & 0 & a_{1''}c_{1''} - a_4 c_{1'} \\ -s_{1'} & -c_{1'} & 0 & a_{1''}s_{1''} - a_4 s_{1'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.9 Kinematics of Typical Manipulator Structures

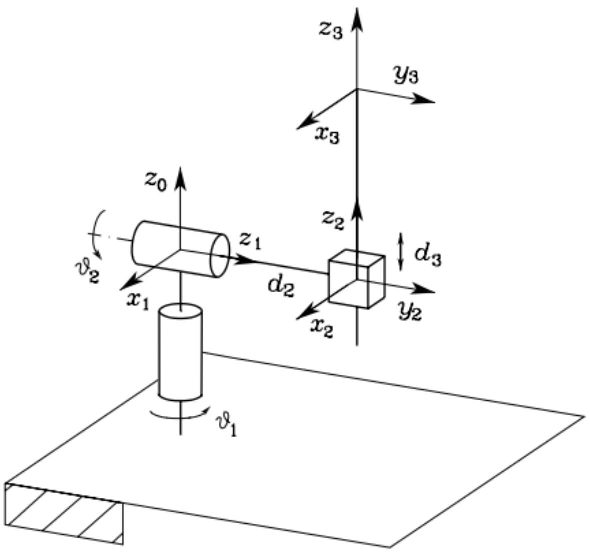
- 2.9.3 Spherical Arm



2.9 Kinematics of Typical Manipulator Structures

- 2.9.3 Spherical Arm
  - DH Parameters

Link	a_i	α_i	d_i	ϑ_i
1	0	$-\pi/2$	0	ϑ_1
2	0	$\pi/2$	d_2	ϑ_2
3	0	0	d_3	0



2.9 Kinematics of Typical Manipulator Structures

- 2.9.3 Spherical Arm
  - The homogeneous transformation matrices:

$$A_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A_2^1(\vartheta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

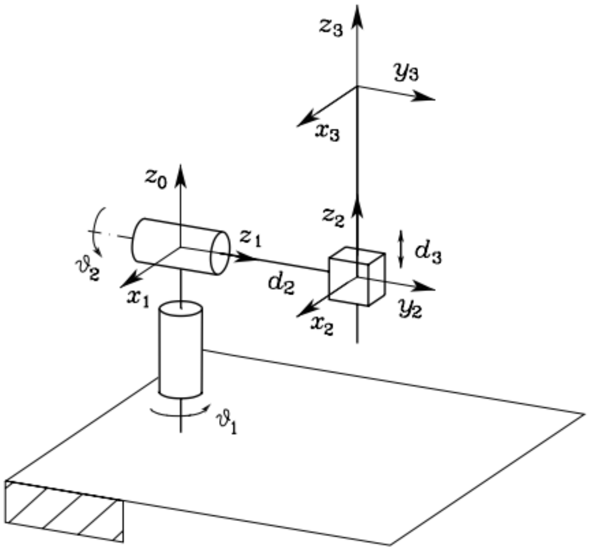
$$A_3^2(d_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2.9 Kinematics of Typical Manipulator Structures

- 2.9.3 Spherical Arm
  - The direct kinematics function:

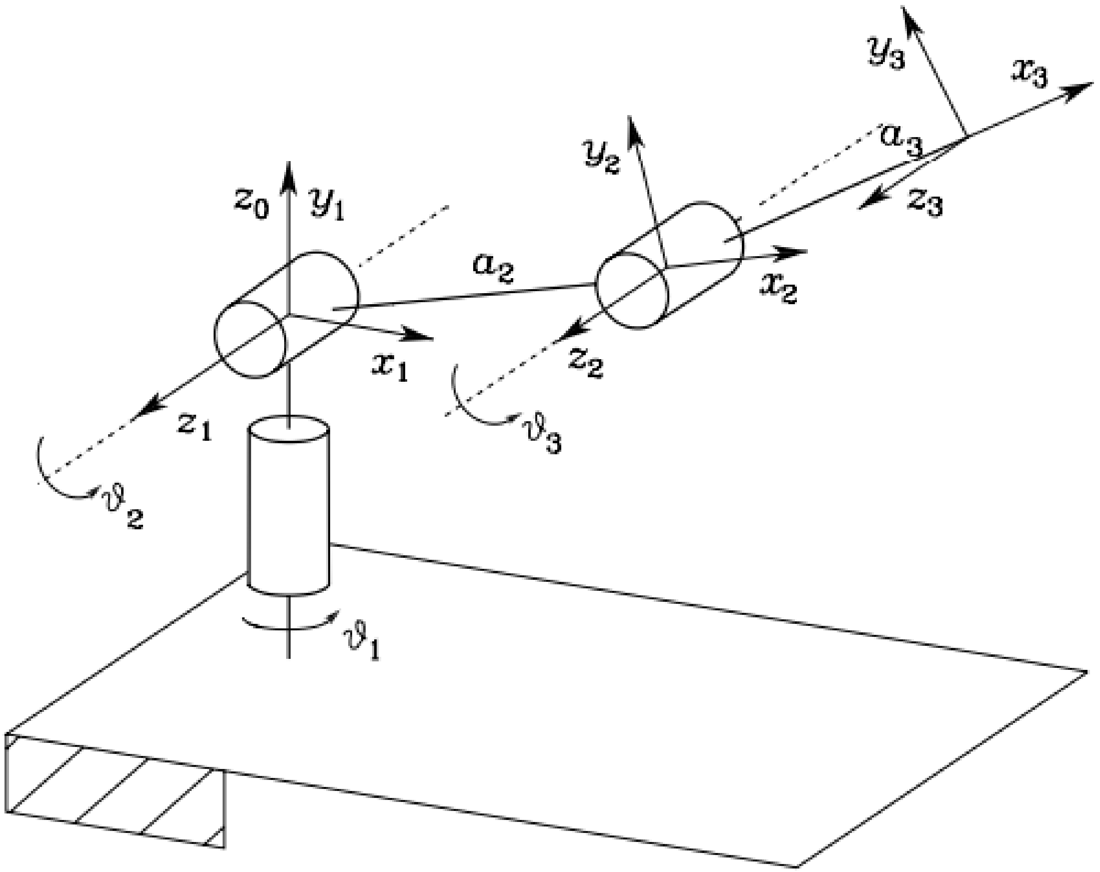
$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0\boldsymbol{A}_2^1\boldsymbol{A}_3^2 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1s_2d_3 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1s_2d_3 + c_1d_2 \\ -s_2 & 0 & c_2 & c_2d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{q} = [\vartheta_1 \quad \vartheta_2 \quad d_3]^T$$



2.9 Kinematics of Typical Manipulator Structures

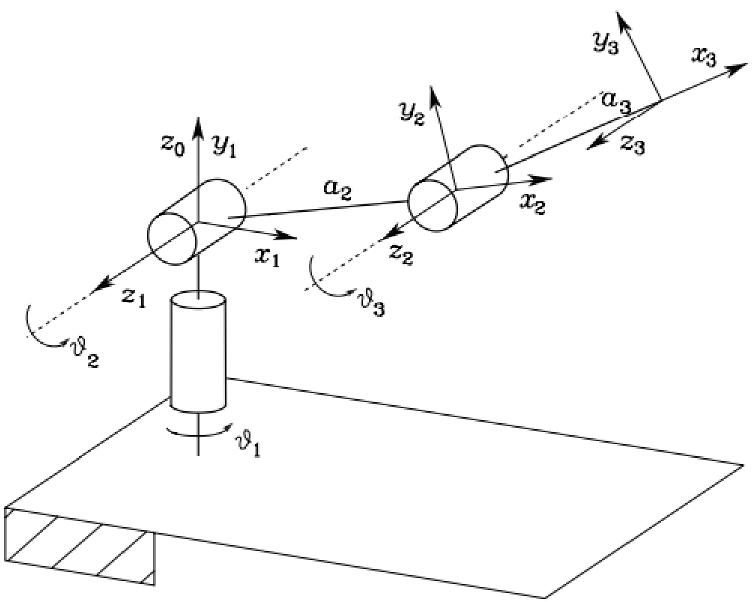
- 2.9.4 Anthropomorphic Arm



2.9 Kinematics of Typical Manipulator Structures

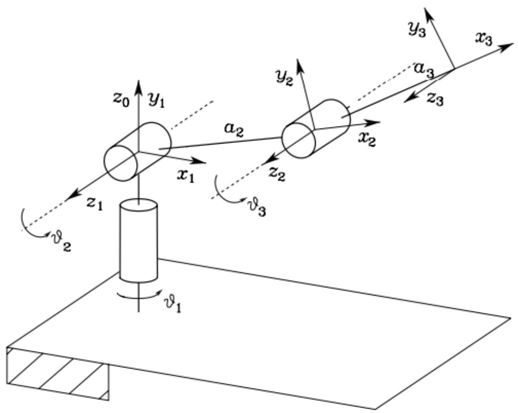
- 2.9.4 Anthropomorphic Arm
  - DH Parameters:

Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	a_3	0	0	ϑ_3



2.9 Kinematics of Typical Manipulator Structures

- 2.9.4 Anthropomorphic Arm



$$\boldsymbol{A}_1^0(\vartheta_1) = \begin{bmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

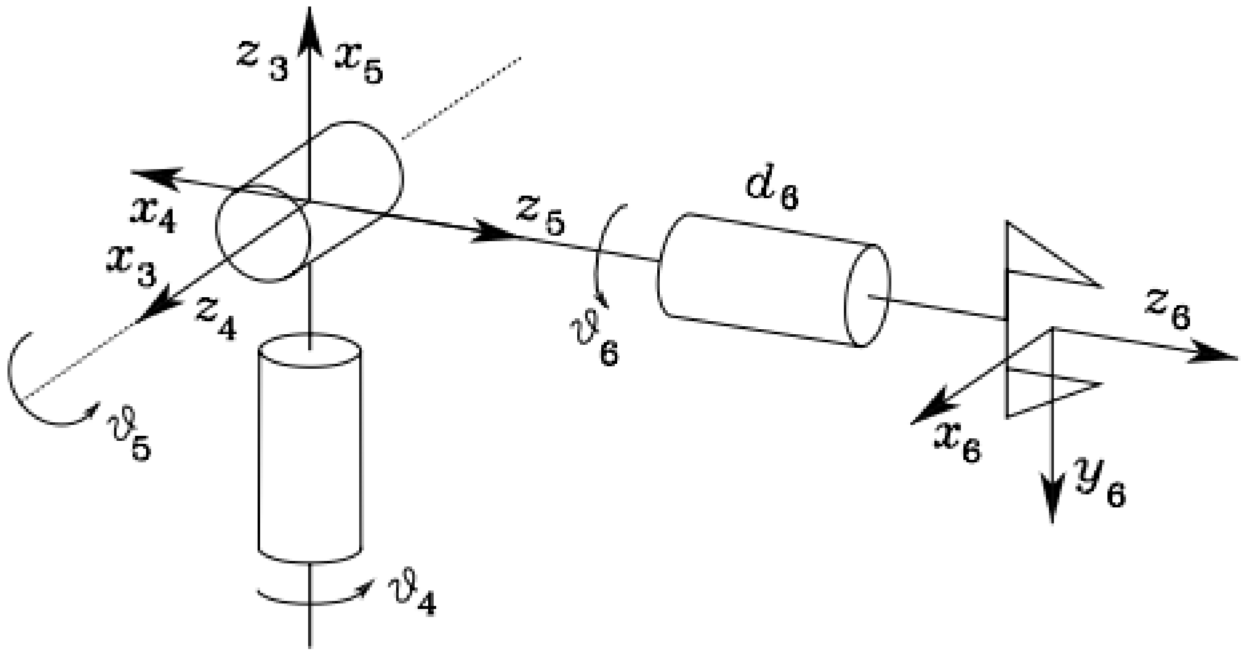
$$\boldsymbol{A}_i^{i-1}(\vartheta_i) = \begin{bmatrix} c_i & -s_i & 0 & a_i c_i \\ s_i & c_i & 0 & a_i s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 2, 3$$

$$\boldsymbol{T}_3^0(\boldsymbol{q}) = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \boldsymbol{A}_3^2 = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & c_1(a_2 c_2 + a_3 c_{23}) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_2 c_2 + a_3 c_{23}) \\ s_{23} & c_{23} & 0 & a_2 s_2 + a_3 s_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{q} = [\vartheta_1 \quad \vartheta_2 \quad \vartheta_3]^T$$

2.9 Kinematics of Typical Manipulator Structures

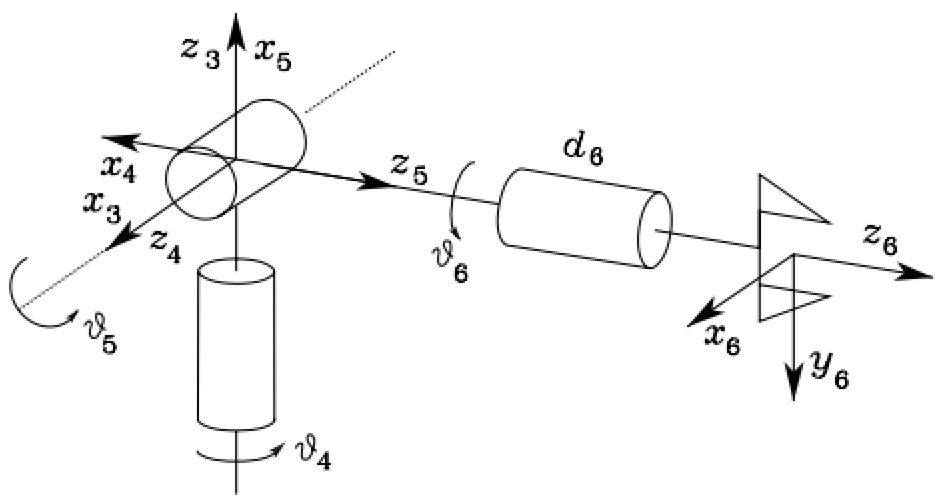
- 2.9.5 Spherical Wrist



2.9 Kinematics of Typical Manipulator Structures

- 2.9.5 Spherical Wrist
  - DH Parameters:

Link	a_i	α_i	d_i	ϑ_i
4	0	$-\pi/2$	0	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

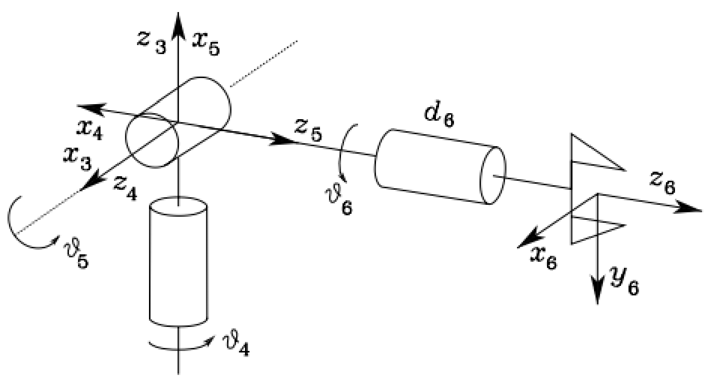


2.9 Kinematics of Typical Manipulator Structures

- 2.9.5 Spherical Wrist

$$\boldsymbol{A}_4^3(\vartheta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{A}_5^4(\vartheta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_6^5(\vartheta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

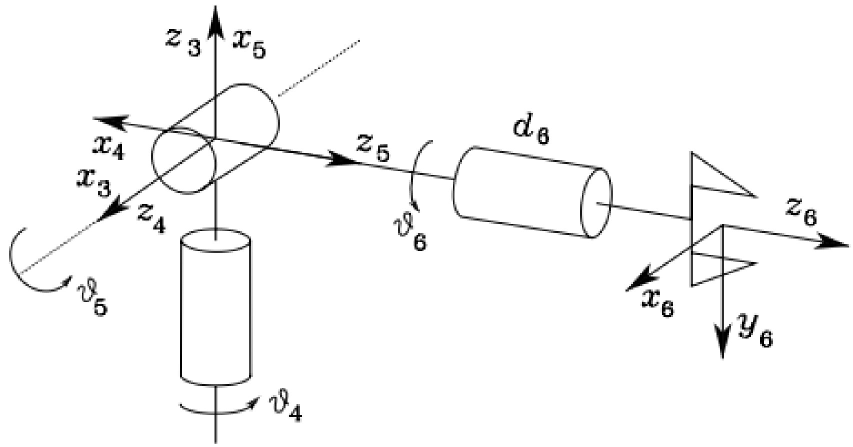


2.9 Kinematics of Typical Manipulator Structures

- 2.9.5 Spherical Wrist

$$\boldsymbol{T}_6^3(\boldsymbol{q}) = \boldsymbol{A}_4^3\boldsymbol{A}_5^4\boldsymbol{A}_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

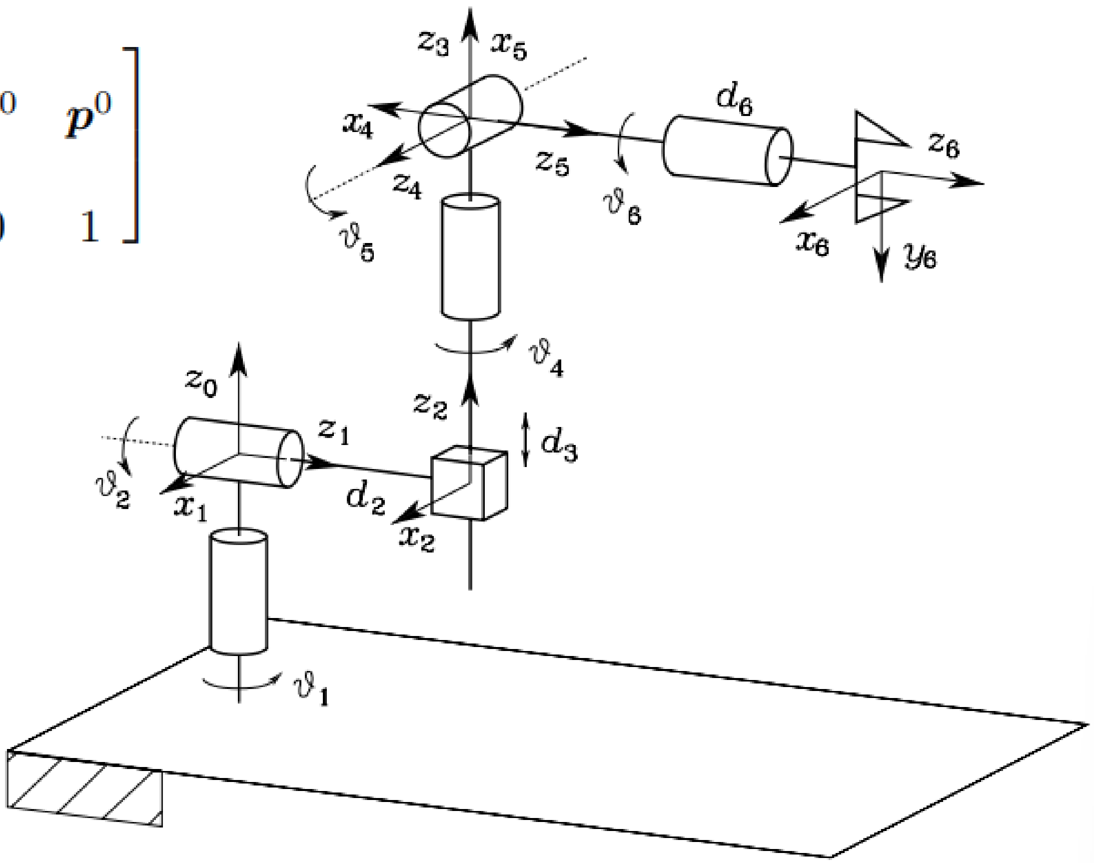
$$\boldsymbol{q} = [\vartheta_4 \quad \vartheta_5 \quad \vartheta_6]^T$$



2.9 Kinematics of Typical Manipulator Structures

- 2.9.6 Stanford Manipulator

$$\boldsymbol{T}_6^0 = \boldsymbol{T}_3^0 \boldsymbol{T}_6^3 = \begin{bmatrix} \boldsymbol{n}^0 & \boldsymbol{s}^0 & \boldsymbol{a}^0 & \boldsymbol{p}^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



2.9 Kinematics of Typical Manipulator Structures

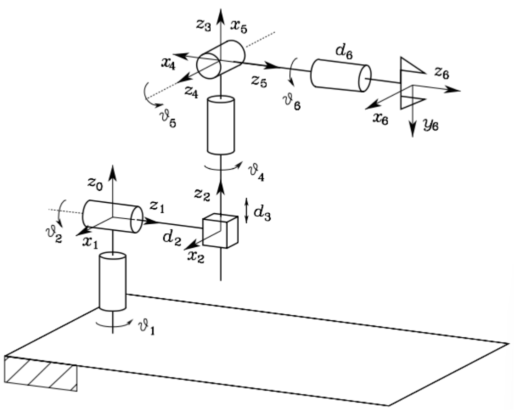
- 2.9.6 Stanford Manipulator

$$p_6^0 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + \big(c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5\big) d_6 \\ s_1 s_2 d_3 + c_1 d_2 + \big(s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5\big) d_6 \\ c_2 d_3 + (-s_2 c_4 s_5 + c_2 c_5) d_6 \end{bmatrix}$$

$$n_6^0 = \begin{bmatrix} c_1\big(c_2(c_4 c_5 c_6 - s_4 s_6) - s_2 s_5 c_6\big) - s_1(s_4 c_5 c_6 + c_4 s_6) \\ s_1\big(c_2(c_4 c_5 c_6 - s_4 s_6) - s_2 s_5 c_6\big) + c_1(s_4 c_5 c_6 + c_4 s_6) \\ -s_2(c_4 c_5 c_6 - s_4 s_6) - c_2 s_5 c_6 \end{bmatrix}$$

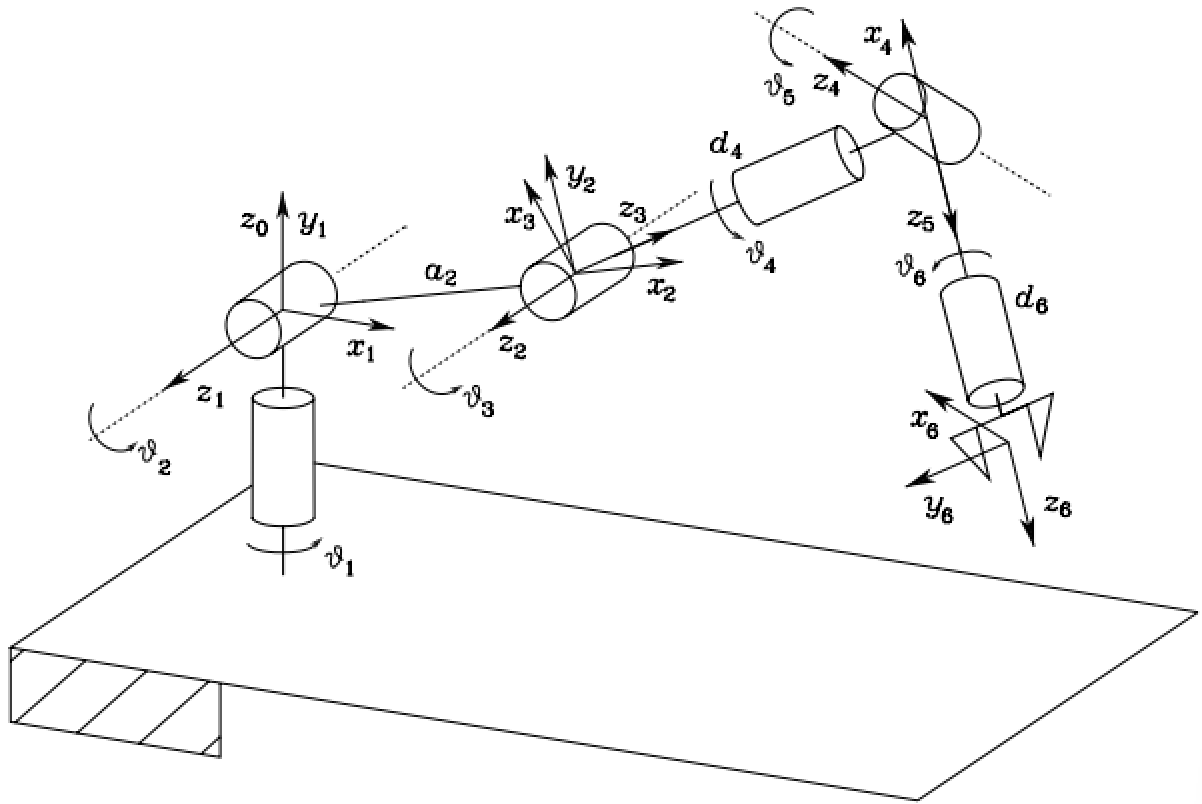
$$s_6^0 = \begin{bmatrix} c_1\big(-c_2(c_4 c_5 s_6 + s_4 c_6) + s_2 s_5 s_6\big) - s_1(-s_4 c_5 s_6 + c_4 c_6) \\ s_1\big(-c_2(c_4 c_5 s_6 + s_4 c_6) + s_2 s_5 s_6\big) + c_1(-s_4 c_5 s_6 + c_4 c_6) \\ s_2(c_4 c_5 s_6 + s_4 c_6) + c_2 s_5 s_6 \end{bmatrix}$$

$$a_6^0 = \begin{bmatrix} c_1(c_2 c_4 s_5 + s_2 c_5) - s_1 s_4 s_5 \\ s_1(c_2 c_4 s_5 + s_2 c_5) + c_1 s_4 s_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$



2.9 Kinematics of Typical Manipulator Structures

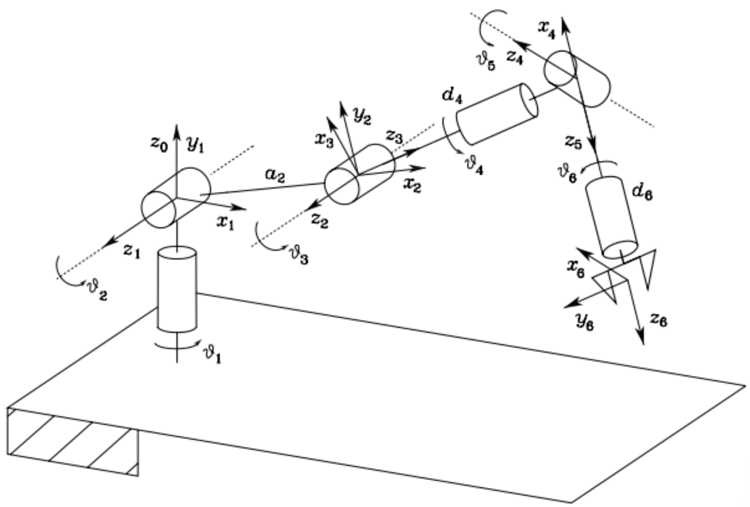
- 2.9.7 Anthropomorphic Arm with Spherical Wrist



2.9 Kinematics of Typical Manipulator Structures

- 2.9.7 Anthropomorphic Arm with Spherical Wrist
  - DH Parameters

Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	a_2	0	0	ϑ_2
3	0	$\pi/2$	0	ϑ_3
4	0	$-\pi/2$	d_4	ϑ_4
5	0	$\pi/2$	0	ϑ_5
6	0	0	d_6	ϑ_6

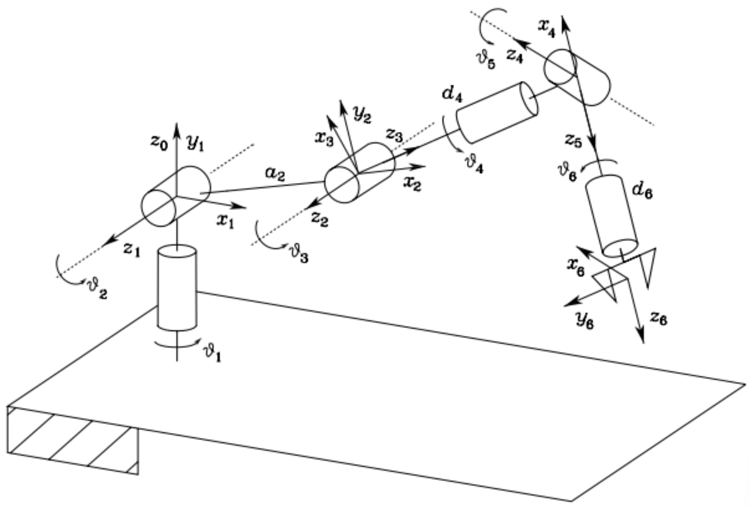


2.9 Kinematics of Typical Manipulator Structures

- 2.9.7 Anthropomorphic Arm with Spherical Wrist

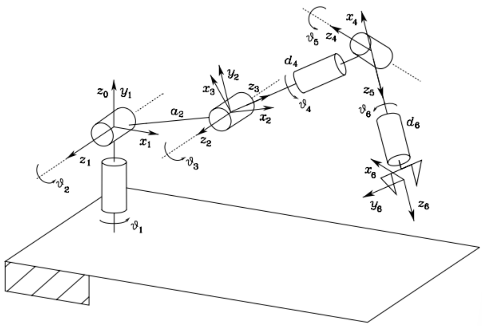
$$\boldsymbol{A}_3^2(\vartheta_3) = \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ s_3 & 0 & -c_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_4^3(\vartheta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



2.9 Kinematics of Typical Manipulator Structures

- 2.9.7 Anthropomorphic Arm with Spherical Wrist



$$\boldsymbol{p}_6^0 = \begin{bmatrix} a_2c_1c_2 + d_4c_1s_{23} + d_6\big(c_1(c_{23}c_4s_5 + s_{23}c_5) + s_1s_4s_5\big) \\ a_2s_1c_2 + d_4s_1s_{23} + d_6\big(s_1(c_{23}c_4s_5 + s_{23}c_5) - c_1s_4s_5\big) \\ a_2s_2 - d_4c_{23} + d_6(s_{23}c_4s_5 - c_{23}c_5) \end{bmatrix}$$

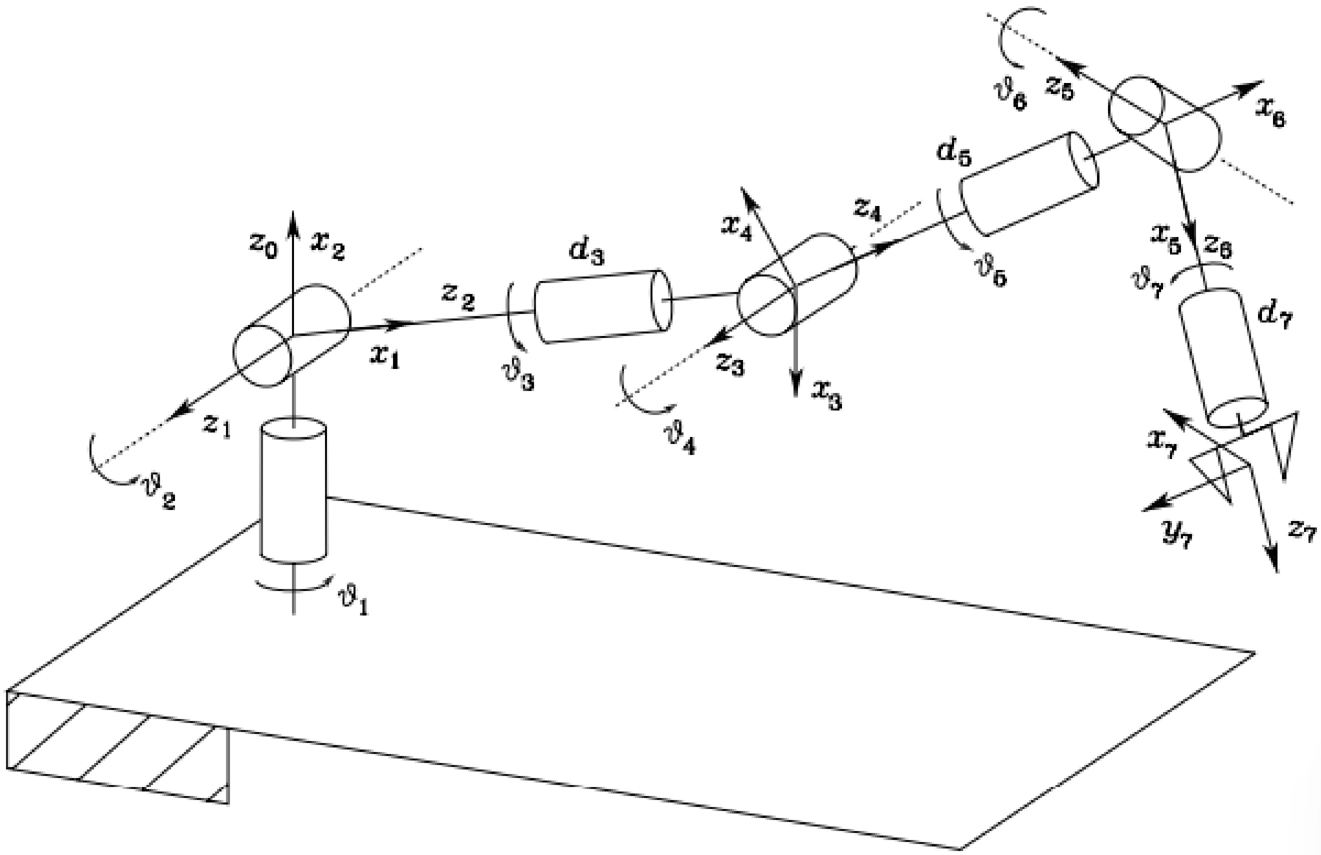
$$\boldsymbol{n}_6^0 = \begin{bmatrix} c_1\big(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\big) + s_1(s_4c_5c_6 + c_4s_6) \\ s_1\big(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\big) - c_1(s_4c_5c_6 + c_4s_6) \\ s_{23}(c_4c_5c_6 - s_4s_6) + c_{23}s_5c_6 \end{bmatrix}$$

$$\boldsymbol{s}_6^0 = \begin{bmatrix} c_1\big(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6\big) + s_1(-s_4c_5s_6 + c_4c_6) \\ s_1\big(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6\big) - c_1(-s_4c_5s_6 + c_4c_6) \\ -s_{23}(c_4c_5s_6 + s_4c_6) - c_{23}s_5s_6 \end{bmatrix}$$

$$\boldsymbol{a}_6^0 = \begin{bmatrix} c_1(c_{23}c_4s_5 + s_{23}c_5) + s_1s_4s_5 \\ s_1(c_{23}c_4s_5 + s_{23}c_5) - c_1s_4s_5 \\ s_{23}c_4s_5 - c_{23}c_5 \end{bmatrix}.$$

2.9 Kinematics of Typical Manipulator Structures

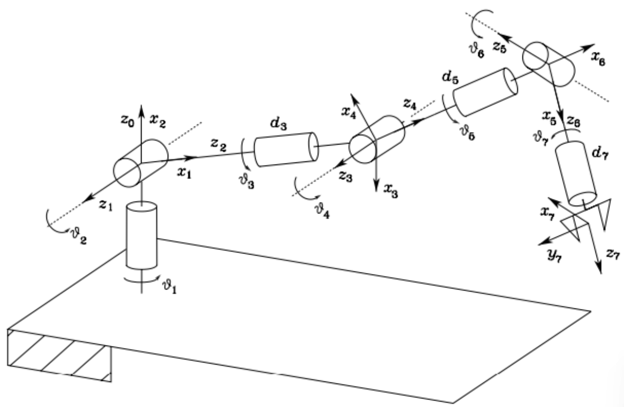
- 2.9.8 DLR Manipulator



2.9 Kinematics of Typical Manipulator Structures

- 2.9.8 DLR Manipulator
  - DH Parameters

Link	a_i	α_i	d_i	ϑ_i
1	0	$\pi/2$	0	ϑ_1
2	0	$\pi/2$	0	ϑ_2
3	0	$\pi/2$	d_3	ϑ_3
4	0	$\pi/2$	0	ϑ_4
5	0	$\pi/2$	d_5	ϑ_5
6	0	$\pi/2$	0	ϑ_6
7	0	0	d_7	ϑ_7

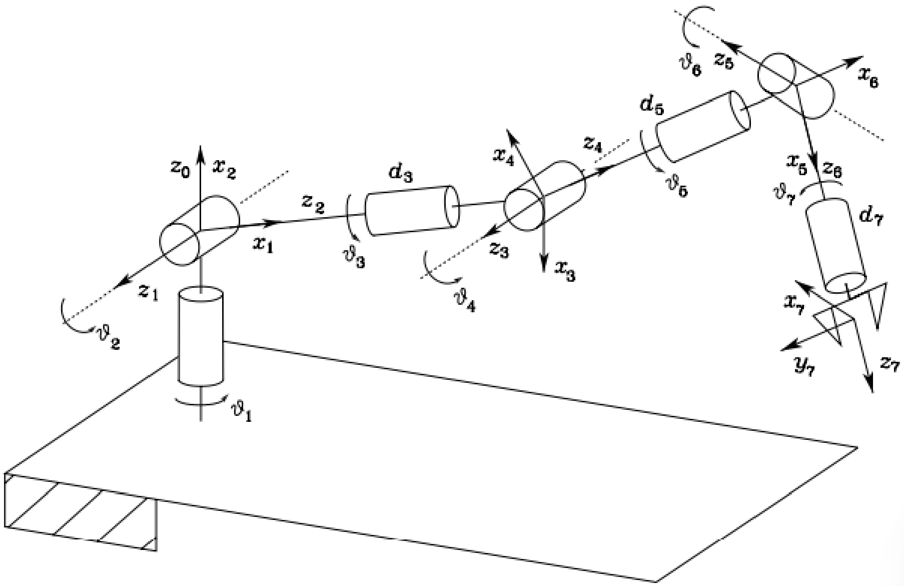


2.9 Kinematics of Typical Manipulator Structures

- 2.9.8 DLR Manipulator

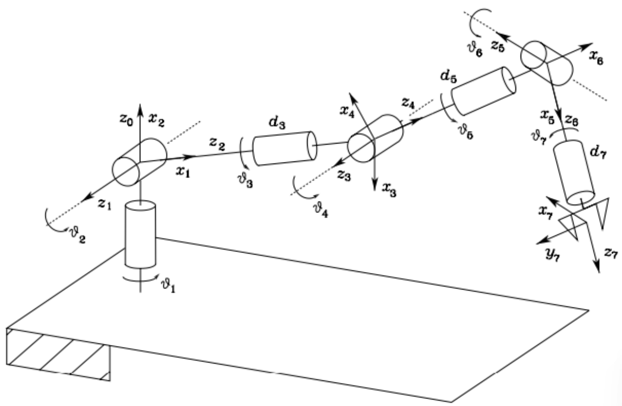
$$\boldsymbol{A}_i^{i-1} = \begin{bmatrix} c_i & 0 & s_i & 0 \\ s_i & 0 & -c_i & 0 \\ 0 & 1 & 0 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i = 1, \dots, 6$$

$$\boldsymbol{A}_7^6 = \begin{bmatrix} c_7 & -s_7 & 0 & 0 \\ s_7 & c_7 & 0 & 0 \\ 0 & 0 & 1 & d_7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



2.9 Kinematics of Typical Manipulator Structures

- 2.9.8 DLR Manipulator



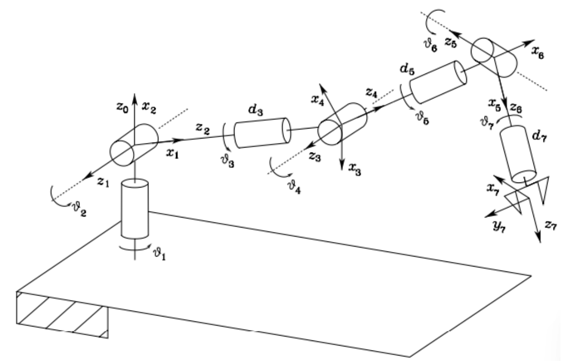
$$\boldsymbol{p}_7^0 = \begin{bmatrix} d_3 x_{d_3} + d_5 x_{d_5} + d_7 x_{d_7} \\ d_3 y_{d_3} + d_5 y_{d_5} + d_7 y_{d_7} \\ d_3 z_{d_3} + d_5 z_{d_5} + d_7 z_{d_7} \end{bmatrix}$$

$$\begin{aligned}
x_{d_3} &= c_1 s_2 \\
x_{d_5} &= c_1(c_2 c_3 s_4 - s_2 c_4) + s_1 s_3 s_4 \\
x_{d_7} &= c_1(c_2 k_1 + s_2 k_2) + s_1 k_3 \\
y_{d_3} &= s_1 s_2 \\
y_{d_5} &= s_1(c_2 c_3 s_4 - s_2 c_4) - c_1 s_3 s_4 \\
y_{d_7} &= s_1(c_2 k_1 + s_2 k_2) - c_1 k_3 \\
z_{d_3} &= -c_2 \\
z_{d_5} &= c_2 c_4 + s_2 c_3 s_4 \\
z_{d_7} &= s_2(c_3(c_4 c_5 s_6 - s_4 c_6) + s_3 s_5 s_6) - c_2 k_2
\end{aligned}$$

$$\begin{aligned}
k_1 &= c_3(c_4 c_5 s_6 - s_4 c_6) + s_3 s_5 s_6 \\
k_2 &= s_4 c_5 s_6 + c_4 c_6 \\
k_3 &= s_3(c_4 c_5 s_6 - s_4 c_6) - c_3 s_5 s_6
\end{aligned}$$

2.9 Kinematics of Typical Manipulator Structures

- 2.9.8 DLR Manipulator



$$\boldsymbol{n}_7^0 = \begin{bmatrix} ((x_a c_5 + x_c s_5)c_6 + x_b s_6)c_7 + (x_a s_5 - x_c c_5)s_7 \\ ((y_a c_5 + y_c s_5)c_6 + y_b s_6)c_7 + (y_a s_5 - y_c c_5)s_7 \\ (z_a c_6 + z_c s_6)c_7 + z_b s_7 \end{bmatrix}$$

$$\boldsymbol{s}_7^0 = \begin{bmatrix} -((x_a c_5 + x_c s_5)c_6 + x_b s_6)s_7 + (x_a s_5 - x_c c_5)c_7 \\ -((y_a c_5 + y_c s_5)c_6 + y_b s_6)s_7 + (y_a s_5 - y_c c_5)c_7 \\ -(z_a c_6 + z_c s_6)s_7 + z_b c_7 \end{bmatrix}$$

$$\boldsymbol{a}_7^0 = \begin{bmatrix} (x_a c_5 + x_c s_5)s_6 - x_b c_6 \\ (y_a c_5 + y_c s_5)s_6 - y_b c_6 \\ z_a s_6 - z_c c_6 \end{bmatrix},$$

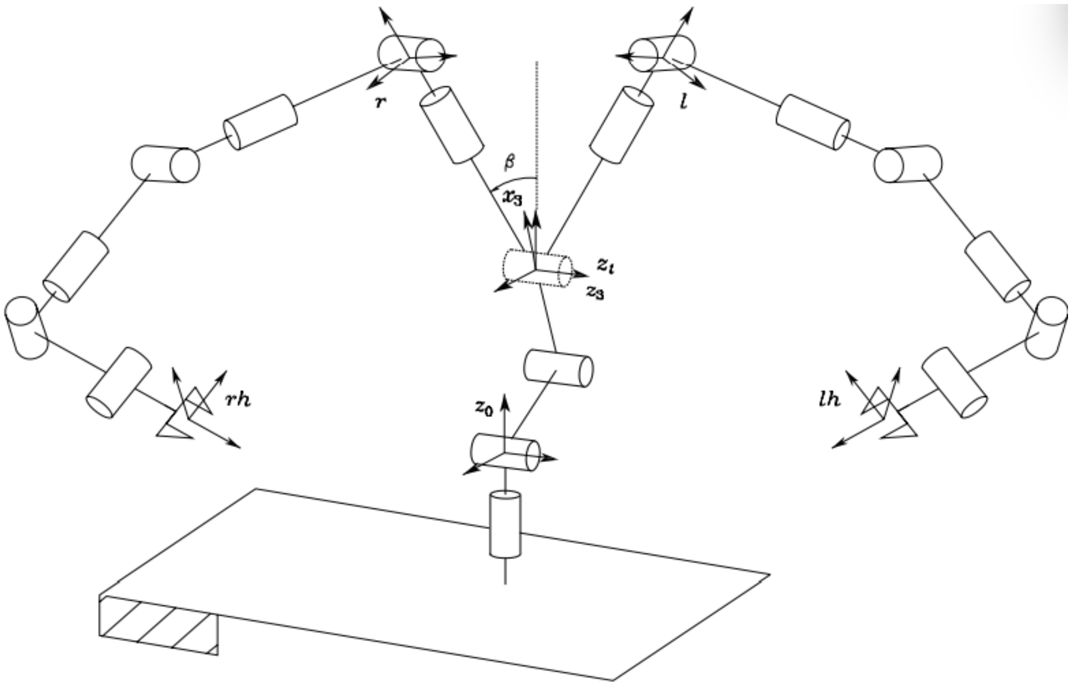
$$\begin{aligned}
x_a &= (c_1 c_2 c_3 + s_1 s_3)c_4 + c_1 s_2 s_4 \\
x_b &= (c_1 c_2 c_3 + s_1 s_3)s_4 - c_1 s_2 c_4 \\
x_c &= c_1 c_2 s_3 - s_1 c_3 \\
y_a &= (s_1 c_2 c_3 - c_1 s_3)c_4 + s_1 s_2 s_4 \\
y_b &= (s_1 c_2 c_3 - c_1 s_3)s_4 - s_1 s_2 c_4 \\
y_c &= s_1 c_2 s_3 + c_1 c_3 \\
z_a &= (s_2 c_3 c_4 - c_2 s_4)c_5 + s_2 s_3 s_5 \\
z_b &= (s_2 c_3 s_4 + c_2 c_4)s_5 - s_2 s_3 c_5 \\
z_c &= s_2 c_3 s_4 + c_2 c_4 .
\end{aligned}$$

2.10 Joint Space and Operational Space

- 2.9.9 Humanoid Manipulator

$$T^0_{rh} = T^0_3\, T^3_t\, T^t_r T^r_{rh}$$
$$T^0_{lh} = T^0_3\, T^3_t\, T^t_l T^l_{lh}$$

$$T^3_t = \begin{bmatrix} c_{23} & s_{23} & 0 & 0 \\ -s_{23} & c_{23} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



2.10 Joint Space and Operational Space

- Direct Kinematics:
  - Position and orientation of the end-effector frame to be expressed as a function of the joint variables with respect to the base frame.
  - This is quite easy for the position, but quite difficult for orientation
    (9 components must be guaranteed to satisfy the orthonormality constraints)

- The end-effector pose can be given by a minimal number of coordinates and minimal representation (Euler angles) describing the rotation

$$\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{p}_e \\ \boldsymbol{\phi}_e \end{bmatrix}$$

  - p_e : End-effector position
  - φ_e : End-effector orientation

2.10 Joint Space and Operational Space

- The vector x_e is defined in the space in which the manipulator task is specified; hence, this space is typically called ***operational space***.

$$\boldsymbol{x}_e = \begin{bmatrix} \boldsymbol{p}_e \\ \boldsymbol{\phi}_e \end{bmatrix}$$

- On the other hand, the ***joint space*** (configuration space) denotes the space in which the (n×1) vector of joint variables

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

  - For a revolute joint: $q_i = \vartheta_i$
  - For a prismatic joint: $q_i = d_i$

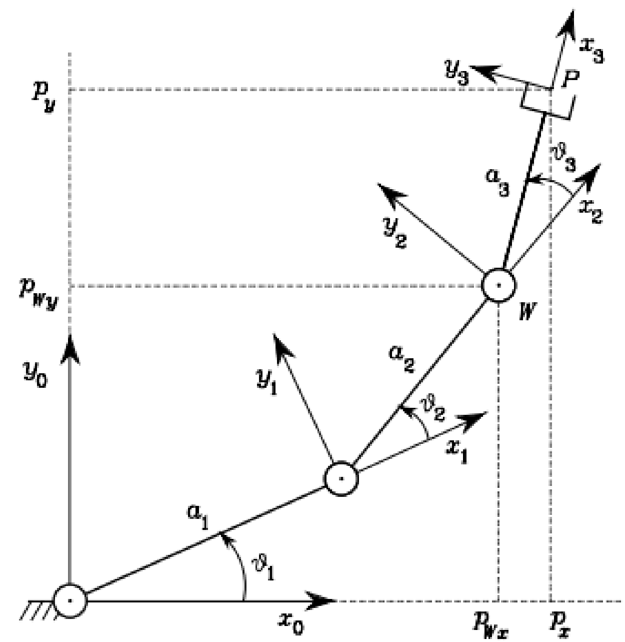
- ***Direct Kinematics Equation:*** $\boldsymbol{x}_e = \boldsymbol{k}(\boldsymbol{q})$

2.10 Joint Space and Operational Space

- Example 2.5

$$\boldsymbol{x}_e = \begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \boldsymbol{k}(\boldsymbol{q}) = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

- 3 joint space variables allow specification of at most 3 independent operational space variables.
- If orientation is of no concern, there is *kinematic redundancy.*



2.10 Joint Space and Operational Space

- 2.10.1 Workspace
- ***Workspace:***

  The region described by the origin of the end-effector frame when all the manipulator joints execute all possible motions

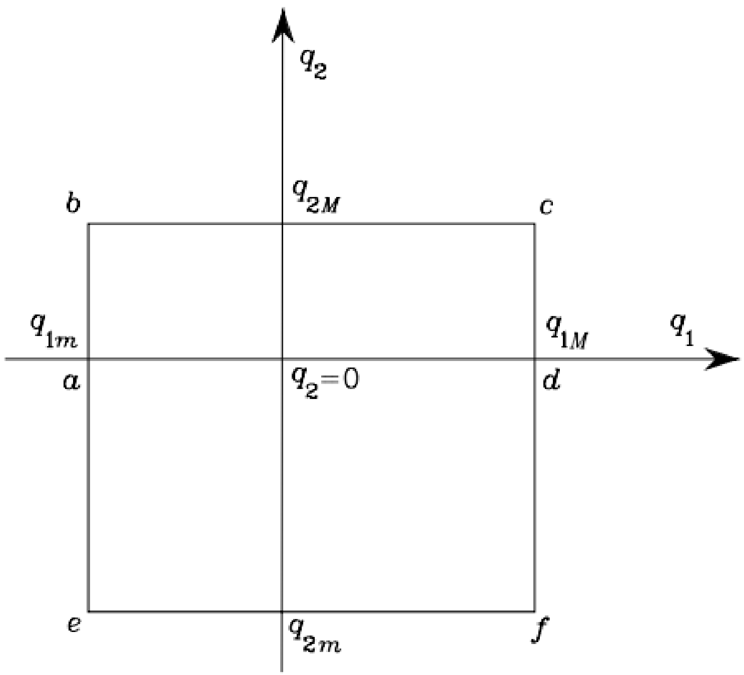
  - *Reachable workspace*
  - *Dexterous workspace*

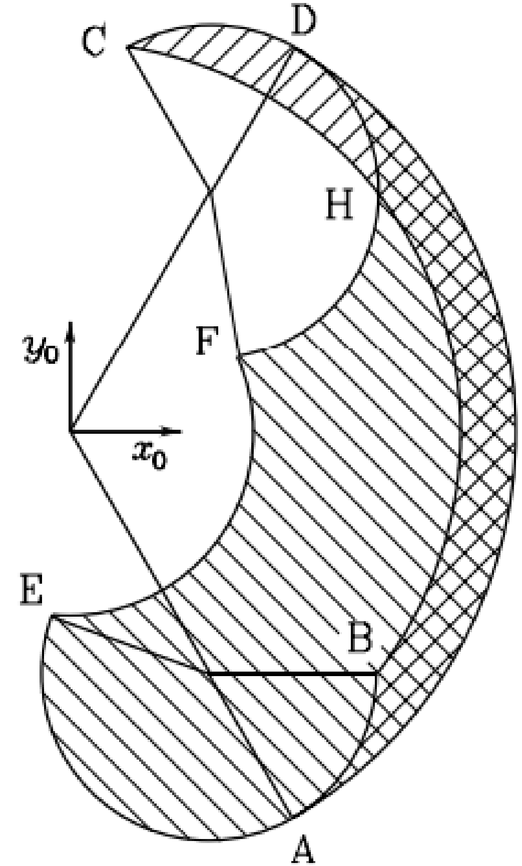
$$\boldsymbol{p}_e = \boldsymbol{p}_e(\boldsymbol{q}) \qquad\qquad q_{im} \leq q_i \leq q_{iM} \quad i = 1, \ldots, n$$

- This volume is finite, closed, connected and is defined by its bordering surface.

2.10 Joint Space and Operational Space

- Example 2.6
  - The simple two-link planar arm





2.10 Joint Space and Operational Space

- 2.10.2 Kinematic Redundancy
- Kinematically Redundant:
  - When number of DOFs is greater than the number of variables that are necessary to describe a given task

- A manipulator is intrinsically redundant when the dimension of the operational space is smaller than the dimension of the joint space ($m < n$)

- Redundancy is a concept relative to the task assigned to the manipulator.

2.11 Kinematic Calibration

- The Denavit – Hartenberg parameters for direct kinematics need to be computed as precisely as possible in order to improve manipulator accuracy.

- Kinematic calibration techniques are devoted to finding accurate estimates of DH parameters from a series of measurements on the manipulator’s end-effector pose.

- As a result of the kinematic calibration procedure, more accurate estimates of the real manipulator geometric parameters as well as possible corrections to make on the joint transducers measurements are obtained.

2.12 Inverse Kinematics Problem

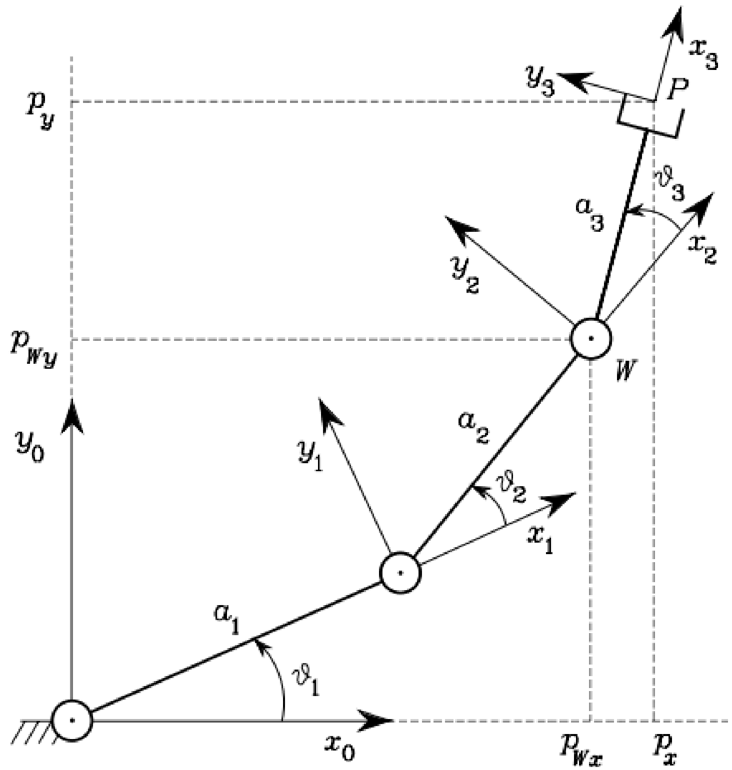
- The inverse kinematics problem consists of the determination of the joint variables corresponding to a given end-effector position and orientation.

- It transforms the motion specifications, assigned to the end-effector in the operational space, into the corresponding joint space motions that allow execution of the desired motion.

- The inverse kinematics problem is much more complex:
  - The equations to solve are in general nonlinear
  - Multiple solutions may exist
  - Infinite solutions may exist
  - There might be no admissible solutions

2.12 Inverse Kinematics Problem

- 2.12.1 Solution of Three-link Planar Arm
  - The end-effector position and orientation in terms of a minimal number of parameters:
    - ✓ The two coordinates p_x, p_y
    - ✓ The angle φ with axis x_0



2.12 Inverse Kinematics Problem

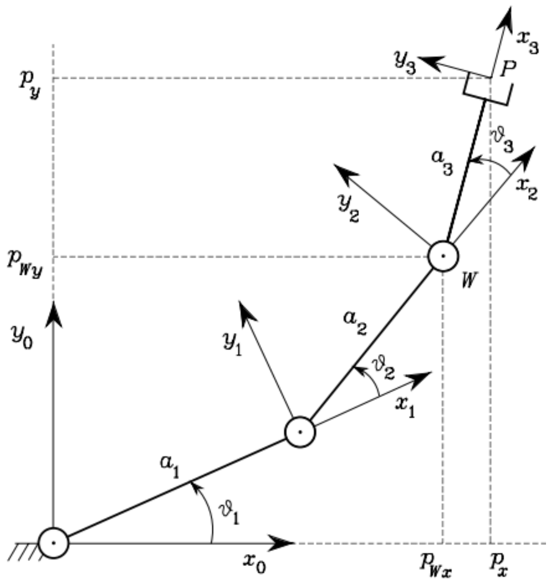
- 2.12.1 Solution of Three-link Planar Arm
  - *Algebraic solution technique*

$$\phi = \vartheta_1 + \vartheta_2 + \vartheta_3$$

$$p_{Wx} = p_x - a_3 c_\phi = a_1 c_1 + a_2 c_{12}$$
$$p_{Wy} = p_y - a_3 s_\phi = a_1 s_1 + a_2 s_{12}$$

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 + 2 a_1 a_2 c_2$$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2 a_1 a_2}$$



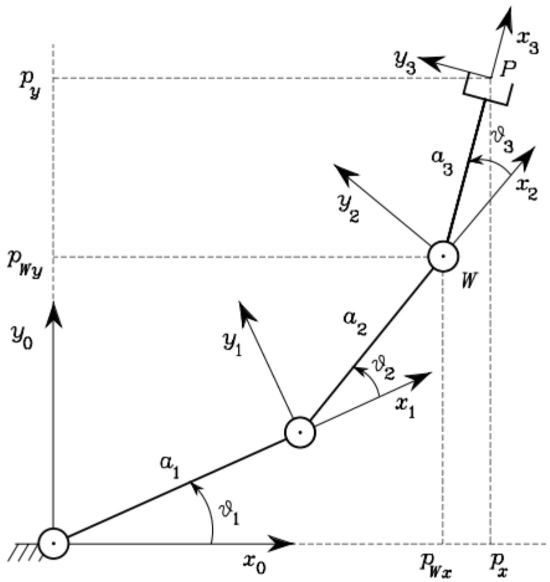
2.12 Inverse Kinematics Problem

- 2.12.1 Solution of Three-link Planar Arm
  - *Algebraic solution technique*

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1a_2}$$

$$s_2 = \pm\sqrt{1 - c_2^2}$$

$$\vartheta_2 = \mathrm{Atan2}(s_2, c_2)$$



2.12 Inverse Kinematics Problem

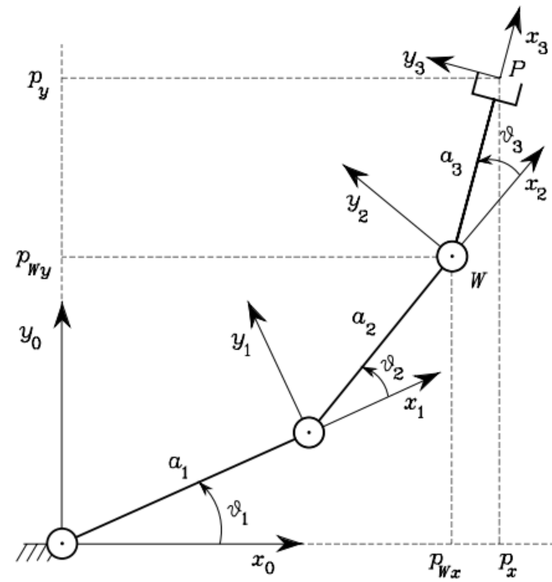
- 2.12.1 Solution of Three-link Planar Arm
  - *Algebraic solution technique*

$$s_1 = \frac{(a_1 + a_2 c_2) p_{Wy} - a_2 s_2 p_{Wx}}{p_{Wx}^2 + p_{Wy}^2}$$

$$c_1 = \frac{(a_1 + a_2 c_2) p_{Wx} + a_2 s_2 p_{Wy}}{p_{Wx}^2 + p_{Wy}^2}$$

$$\vartheta_1 = \text{Atan2}(s_1, c_1)$$

$$\vartheta_3 = \phi - \vartheta_1 - \vartheta_2$$



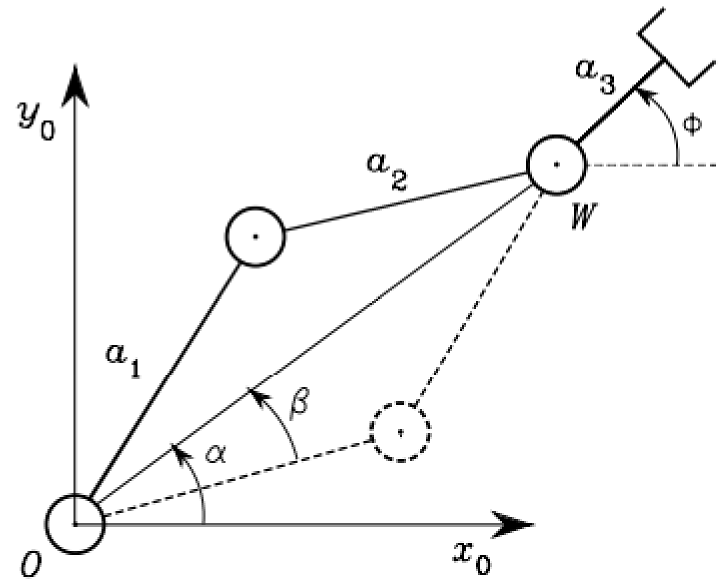
2.12 Inverse Kinematics Problem

- 2.12.1 Solution of Three-link Planar Arm
  - *Geometric solution technique*
    - ✓ The application of the cosine theorem to the triangle formed by links a_1, a_2 and the segment connecting points W and O

$$p_{Wx}^2 + p_{Wy}^2 = a_1^2 + a_2^2 - 2a_1 a_2 \cos(\pi - \vartheta_2)$$

$$c_2 = \frac{p_{Wx}^2 + p_{Wy}^2 - a_1^2 - a_2^2}{2a_1 a_2}$$

$$\vartheta_2 = \pm\cos^{-1}(c_2)$$



- ✓ The elbow-up and elbow-down posture

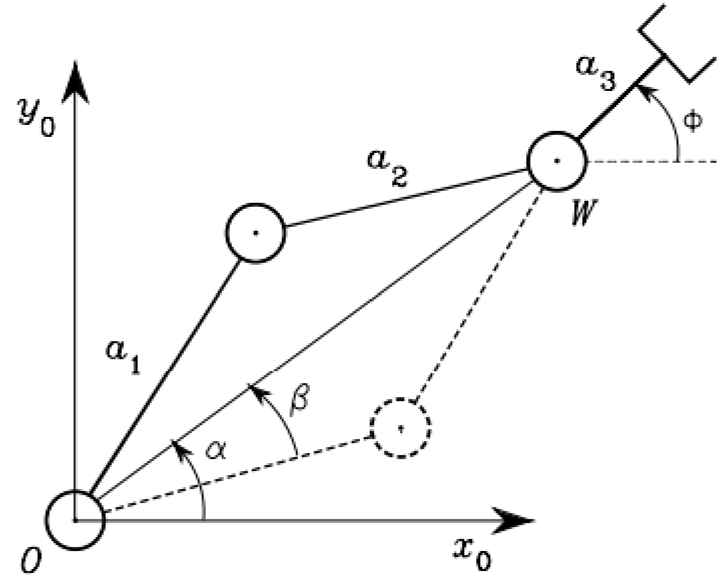
2.12 Inverse Kinematics Problem

- 2.12.1 Solution of Three-link Planar Arm
  - *Geometric solution technique*

$$c_\beta \sqrt{p_{Wx}^2 + p_{Wy}^2} = a_1 + a_2 c_2 \quad \rightarrow \quad \beta = \cos^{-1}\left(\frac{p_{Wx}^2 + p_{Wy}^2 + a_1^2 - a_2^2}{2a_1\sqrt{p_{Wx}^2 + p_{Wy}^2}}\right)$$

$$\alpha = \text{Atan2}(p_{Wy}, p_{Wx})$$

$$\rightarrow \quad \vartheta_1 = \alpha \pm \beta$$



2.12 Inverse Kinematics Problem

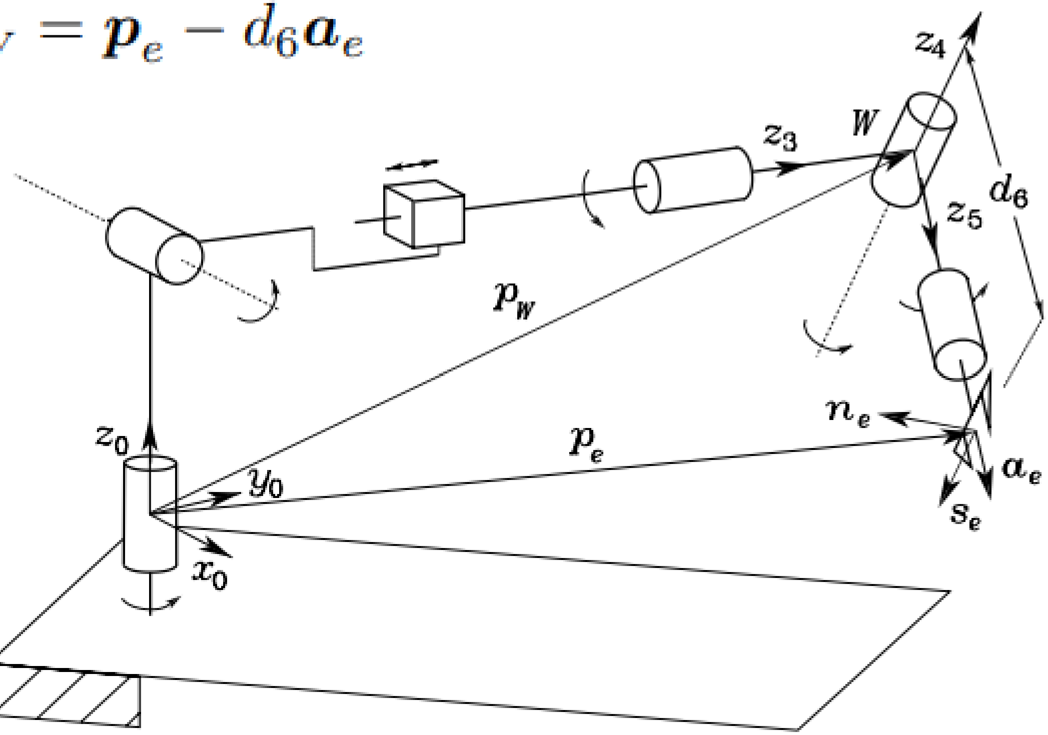
- 2.12.2 Solution of Manipulators with Spherical Wrist
  - Most of the existing manipulators are kinematically simple, since they are typically formed by an arm, of the kind presented above, and a spherical wrist.
  - A suitable point along the structure can be found whose position can be expressed both as a function of the given end-effector position and orientation and as a function of a reduced number of joint variables.
  - This is equivalent to articulating the inverse kinematics problem into two subproblems, since the solution for the position is decoupled from that for the orientation.

2.12 Inverse Kinematics Problem

- 2.12.2 Solution of Manipulators with Spherical Wrist

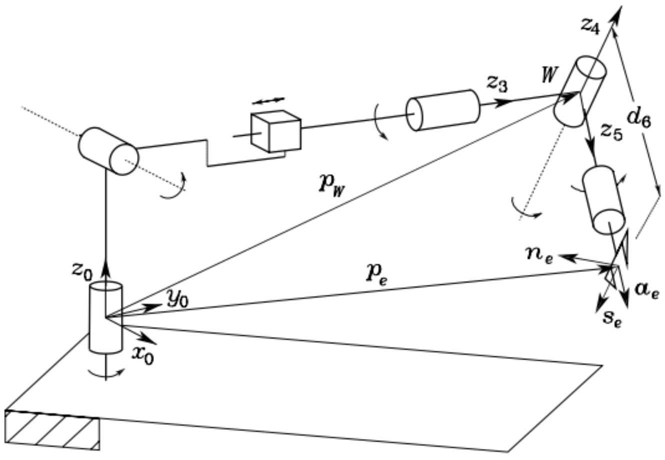
$$\boldsymbol{R}_e = [\,\boldsymbol{n}_e \quad \boldsymbol{s}_e \quad \boldsymbol{a}_e\,]$$

$$\boldsymbol{p}_W = \boldsymbol{p}_e - d_6 \boldsymbol{a}_e$$



2.12 Inverse Kinematics Problem

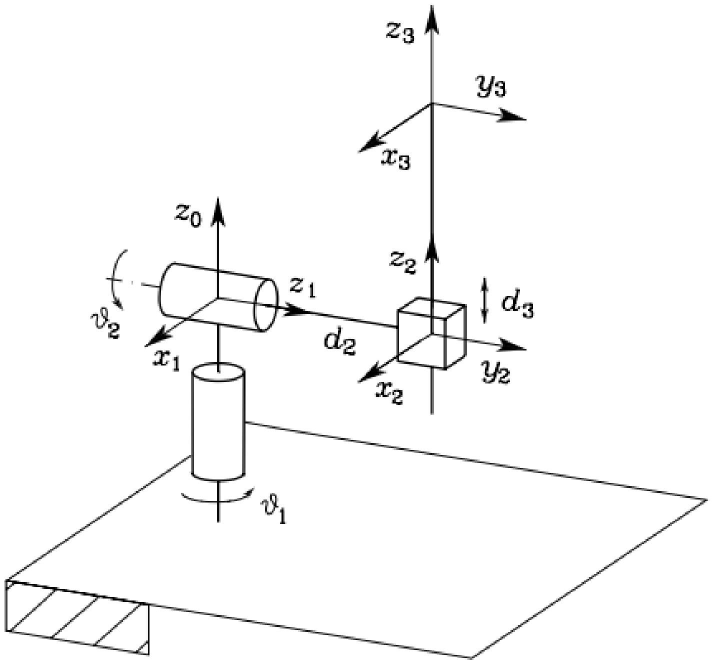
- 2.12.2 Solution of Manipulators with Spherical Wrist
  - The inverse kinematics can be solved according to the following steps:
    - ✓ Compute the wrist position $p_W(q_1, q_2, q_3)$
    - ✓ Solve inverse kinematics for (q_1, q_2, q_3)
    - ✓ Compute $R^0_3(q_1, q_2, q_3)$
    - ✓ Compute $R^3_6(\vartheta_4, \vartheta_5, \vartheta_6) = R^{0\,T}_3 R$
    - ✓ Solve inverse kinematics for orientation $(\vartheta_4, \vartheta_5, \vartheta_6)$



  - It is possible to solve the inverse kinematics for the arm separately from the inverse kinematics for the spherical wrist.

2.12 Inverse Kinematics Problem

- 2.12.3 Solution of Spherical Arm



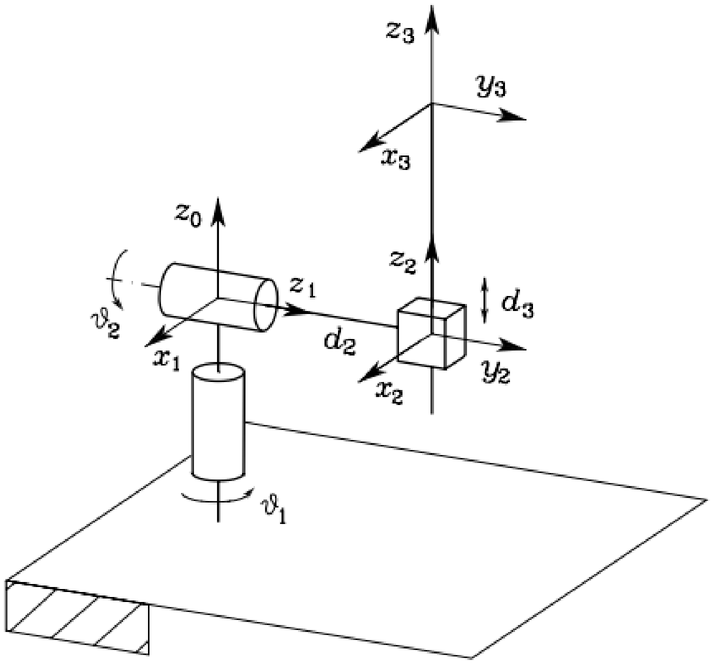
$$\boldsymbol{p}_W^1 = \begin{bmatrix} p_{Wx}c_1 + p_{Wy}s_1 \\ -p_{Wz} \\ -p_{Wx}s_1 + p_{Wy}c_1 \end{bmatrix} = \begin{bmatrix} d_3 s_2 \\ -d_3 c_2 \\ d_2 \end{bmatrix}$$

$$t = \tan\frac{\vartheta_1}{2}$$

$$\rightarrow \quad c_1 = \frac{1-t^2}{1+t^2} \qquad s_1 = \frac{2t}{1+t^2}$$

2.12 Inverse Kinematics Problem

- 2.12.3 Solution of Spherical Arm



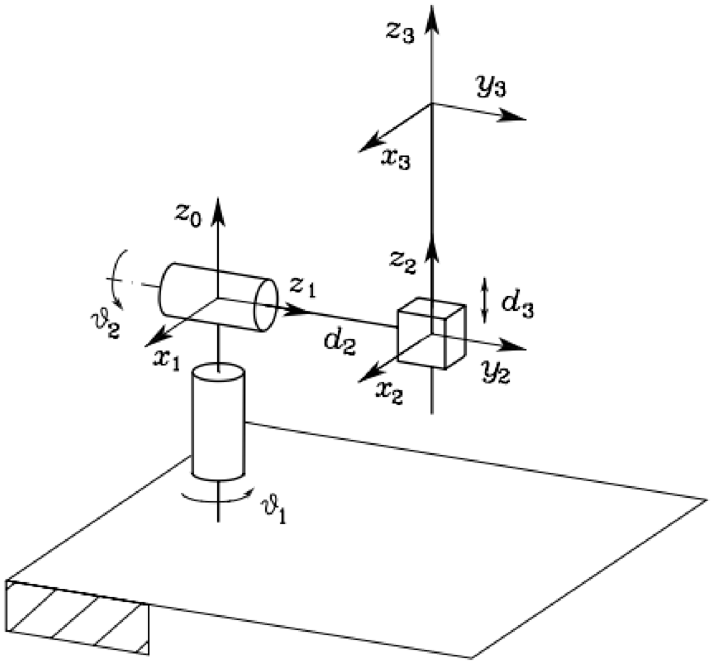
$$(d_2 + p_{Wy})t^2 + 2p_{Wx}t + d_2 - p_{Wy} = 0$$

$$t = \frac{-p_{Wx} \pm \sqrt{p_{Wx}^2 + p_{Wy}^2 - d_2^2}}{d_2 + p_{Wy}}$$

$$\vartheta_1 = 2\text{Atan2}\left(-p_{Wx} \pm \sqrt{p_{Wx}^2 + p_{Wy}^2 - d_2^2},\ d_2 + p_{Wy}\right)$$

2.12 Inverse Kinematics Problem

- 2.12.3 Solution of Spherical Arm



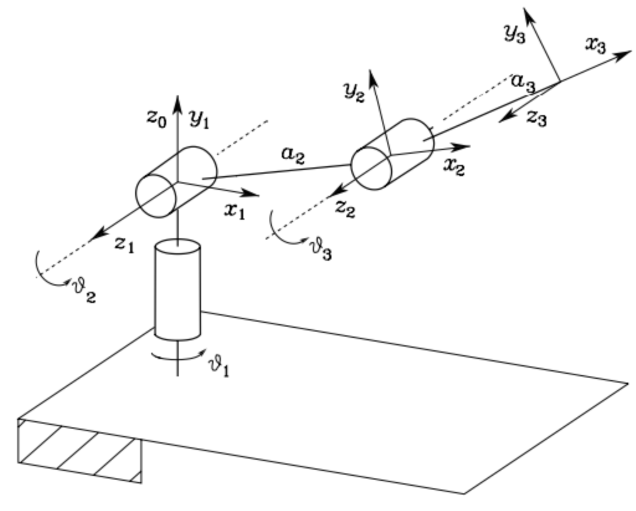
$$d_3 = \sqrt{(p_{Wx}c_1 + p_{Wy}s_1)^2 + p_{Wz}^2}$$

$$\frac{p_{Wx}c_1 + p_{Wy}s_1}{-p_{Wz}} = \frac{d_3 s_2}{-d_3 c_2}$$

$$\vartheta_2 = \text{Atan2}(p_{Wx}c_1 + p_{Wy}s_1, p_{Wz})$$

2.12 Inverse Kinematics Problem

- 2.12.4 Solution of Anthropomorphic Arm



$$
\begin{aligned}
p_{Wx} &= c_1(a_2c_2 + a_3c_{23}) \\
p_{Wy} &= s_1(a_2c_2 + a_3c_{23}) \\
p_{Wz} &= a_2s_2 + a_3s_{23}.
\end{aligned}
$$

$$\rightarrow \quad p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 = a_2^2 + a_3^2 + 2a_2a_3c_3$$

$$\rightarrow \quad c_3 = \frac{p_{Wx}^2 + p_{Wy}^2 + p_{Wz}^2 - a_2^2 - a_3^2}{2a_2a_3}$$

$$\rightarrow \quad s_3 = \pm\sqrt{1 - c_3^2}$$

2.12 Inverse Kinematics Problem

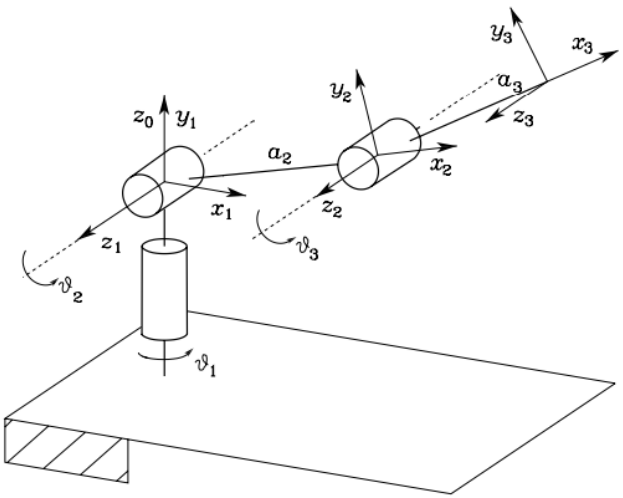
- 2.12.4 Solution of Anthropomorphic Arm

$$\vartheta_3 = \text{Atan2}(s_3, c_3)$$

$$\rightarrow \quad \vartheta_{3,\text{I}} \in [-\pi, \pi] \qquad \vartheta_{3,\text{II}} = -\vartheta_{3,I}.$$

$$p_{Wx}^2 + p_{Wy}^2 = (a_2 c_2 + a_3 c_{23})^2$$

$$\rightarrow \quad a_2 c_2 + a_3 c_{23} = \pm\sqrt{p_{Wx}^2 + p_{Wy}^2}$$



$$\rightarrow \quad c_2 = \frac{\pm\sqrt{p_{Wx}^2 + p_{Wy}^2}(a_2 + a_3 c_3) + p_{Wz} a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3}$$

$$s_2 = \frac{p_{Wz}(a_2 + a_3 c_3) \mp \sqrt{p_{Wx}^2 + p_{Wy}^2}\, a_3 s_3}{a_2^2 + a_3^2 + 2a_2 a_3 c_3}.$$

$$\rightarrow \quad \vartheta_2 = \text{Atan2}(s_2, c_2)$$

2.12 Inverse Kinematics Problem

- 2.12.4 Solution of Anthropomorphic Arm

$$s_3^{+} = \sqrt{1 - c_3^2} \quad \longrightarrow$$

$$\vartheta_{2,\mathrm{I}} = \mathrm{Atan2}\Big((a_2 + a_3 c_3) p_{Wz} - a_3 s_3^{+} \sqrt{p_{Wx}^2 + p_{Wy}^2},\ (a_2 + a_3 c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3 s_3^{+} p_{Wz}\Big)$$

$$\vartheta_{2,\mathrm{II}} = \mathrm{Atan2}\Big((a_2 + a_3 c_3) p_{Wz} + a_3 s_3^{+} \sqrt{p_{Wx}^2 + p_{Wy}^2},\ -(a_2 + a_3 c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3 s_3^{+} p_{Wz}\Big)$$

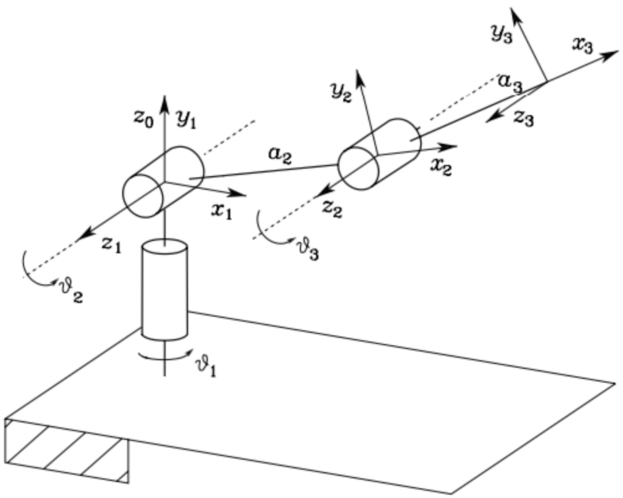
$$s_3^{-} = -\sqrt{1 - c_3^2} \quad \longrightarrow$$

$$\vartheta_{2,\mathrm{III}} = \mathrm{Atan2}\Big((a_2 + a_3 c_3) p_{Wz} - a_3 s_3^{-} \sqrt{p_{Wx}^2 + p_{Wy}^2},\ (a_2 + a_3 c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3 s_3^{-} p_{Wz}\Big)$$

$$\vartheta_{2,\mathrm{IV}} = \mathrm{Atan2}\Big((a_2 + a_3 c_3) p_{Wz} + a_3 s_3^{-} \sqrt{p_{Wx}^2 + p_{Wy}^2},\ -(a_2 + a_3 c_3)\sqrt{p_{Wx}^2 + p_{Wy}^2} + a_3 s_3^{-} p_{Wz}\Big)$$

2.12 Inverse Kinematics Problem

- 2.12.4 Solution of Anthropomorphic Arm



$$p_{Wx} = \pm c_1 \sqrt{p_{Wx}^2 + p_{Wy}^2}$$
$$p_{Wy} = \pm s_1 \sqrt{p_{Wx}^2 + p_{Wy}^2}$$

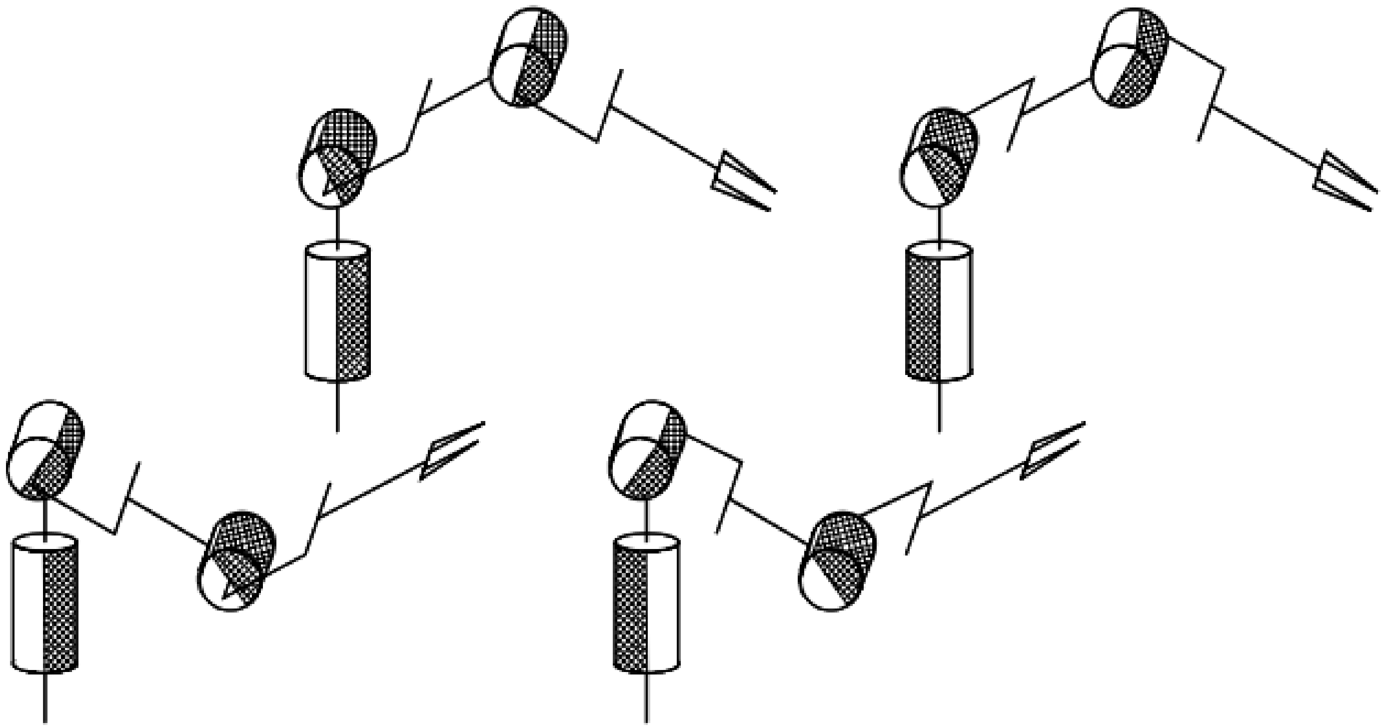
$$\vartheta_{1,\mathrm{I}} = \mathrm{Atan2}(p_{Wy}, p_{Wx})$$
$$\vartheta_{1,\mathrm{II}} = \mathrm{Atan2}(-p_{Wy}, -p_{Wx})$$

$$\vartheta_{1,\mathrm{II}} = \begin{cases} \mathrm{Atan2}(p_{Wy}, p_{Wx}) - \pi & p_{Wy} \geq 0 \\ \mathrm{Atan2}(p_{Wy}, p_{Wx}) + \pi & p_{Wy} < 0 \end{cases}$$

2.12 Inverse Kinematics Problem

- 2.12.4 Solution of Anthropomorphic Arm
  - There exist four solutions:

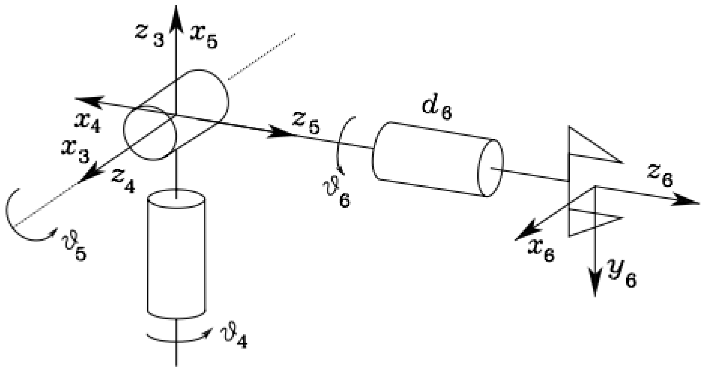
$$(\vartheta_{1,\mathrm{I}}, \vartheta_{2,\mathrm{I}}, \vartheta_{3,\mathrm{I}}) \quad (\vartheta_{1,\mathrm{I}}, \vartheta_{2,\mathrm{III}}, \vartheta_{3,\mathrm{II}}) \quad (\vartheta_{1,\mathrm{II}}, \vartheta_{2,\mathrm{II}}, \vartheta_{3,\mathrm{I}}) \quad (\vartheta_{1,\mathrm{II}}, \vartheta_{2,\mathrm{IV}}, \vartheta_{3,\mathrm{II}})$$



2.12 Inverse Kinematics Problem

- 2.12.5 Solution of Spherical Wrist

$$\boldsymbol{R}_6^3 = \begin{bmatrix} n_x^3 & s_x^3 & a_x^3 \\ n_y^3 & s_y^3 & a_y^3 \\ n_z^3 & s_z^3 & a_z^3 \end{bmatrix}$$



$\vartheta_5 \in (0, \pi)$ →

$$\vartheta_4 = \text{Atan2}(a_y^3, a_x^3)$$
$$\vartheta_5 = \text{Atan2}\left(\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3\right)$$
$$\vartheta_6 = \text{Atan2}(s_z^3, -n_z^3)$$

$\vartheta_5 \in (-\pi, 0)$ →

$$\vartheta_4 = \text{Atan2}(-a_y^3, -a_x^3)$$
$$\vartheta_5 = \text{Atan2}\left(-\sqrt{(a_x^3)^2 + (a_y^3)^2}, a_z^3\right)$$
$$\vartheta_6 = \text{Atan2}(-s_z^3, n_z^3)$$